Construction of Hyperbolic Scarf potential Using GF Technique

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Abstract:
Exact bound state as well as scattering state solutions of Schrodinger Green's function equation for constructing hyperbolic class of exactly solvable quantum system are obtained in D-dimensional space, using extended transformation method. In the current literature the exact solution to the hyperbolic Scarf are written in terms of Jacobi polynomials and these polynomials are referred to Romanovski polynomials. The normalizability of bound state solutions of the generated exactly solvable potential are discussed.
Keywords: Extended transformation, Schrodinger Green's function equation, Exactly solvable potential.

1. 1.Introduction
Exact analytic solution (EAS) of Schrodinger equation with a physical potential is of utmost importance in non-relativistic quantum mechanics (QM). Considerable effort has been made in recent years to obtain the exact solution of the Schrodinger equation for potentials of physical interest [1-9]. However, there is a small set of potentials for which EAS can be found by conventional method. Steiner [10,11] in the course of his work on radial path integral for the first time obtained a general relation between different quantum potentials. His analysis deals with non relativistic space-time transformation of the radial path integral, with a path-dependent change of the time variable. Such Energy-dependent Green's functions (GF) give a general relation between different physical potentials remarkably compact form, when expressed in terms of the radial part of the GF for spherically symmetric potential [12].

In our present work there is no need to invoke any path dependent time transformation for the relation between different potentials. Our analysis would be based on the D-dimensional Schrodinger GF equation. An interesting aspect in this method is that it is not only connection between two 3-dimensional problems, as noted by Steiner. But in our program the dimension of the transformed system we can choose. In this paper we have use a mapping procedure called the extended transformation (ET) [13-18] method, to map already exactly solved quantum system (QS) to new QSs, in any arbitrary D-dimensional Euclidean space, within the framework of the GF technique. We have constructed hyperbolic class of exactly solvable QS considering the trigonometric Scarf potential [19] as input reference potential by implementing the ET method, which is an extension of the $sech^2 r$ potential or original Poschl-Teller potential [20]. The hyperbolic Scarf potential is written in terms of Jacobi polynomials of purely imaginary argument and the parameters are complex conjugate to the trigonometric Scarf potential. These real polynomials are referred as the Ramanovski polynomials [21], which is a family of real orthogonal polynomial and is required to write the exact solution of hyperbolic Scarf potential.

An interesting aspect in the present work is that it gives solution in the region of bound state as well as scattering state but Stenier dealt with only the bound state solutions. It has a wider region of applicability such as linear plus Coulomb potential, non solvable potential of considerable interest in charmonium physics. The Ramanovski polynomials with hyperbolic Scarf potential are required in exact solutions of several physics problems ranging from supper symmetric QM [22,23] and quark physics, to random matrix theory. Another significant point is that the wave- functions of the generated quantum systems are almost normalizable.

The paper is organized as follows. In sec. 2 we present a brief overview of the ET method. In section 3. we make use of ET on Trigonometric Scarf potential to construction new class of exactly solved QSs. Conclusion and finding of our results are discussed in section 4.

3.FORMALISM

For a QS, say A-QS the radial part of the Schrodinger GF equation [13,14] with spherically symmetric potential $V_A(r)$, in D_A-dimensional Euclidean space (in natural units $\hbar = 1 = 2m$) is

$$\left[\partial_r^2 + \frac{D_A - 1}{r}\partial_r + \epsilon_n^A - V_A(r)\right] G_A(r, r_0; \epsilon_n^A) = \frac{\delta(r - r_0)}{r_0^{D_A - 1}} \quad (1)$$

where r is a dimensionless spatial coordinate.

The corresponding integral equation is

$$\varphi_A(r) = \int G_A(r, r_0; \epsilon_n^A)(\epsilon_n^A - V_A(r))\varphi_A(r_0) r_0^{D_A - 1} d\, r_0 \quad (2)$$

where the GF and energy eigenvalues ϵ_n^A are known for the given potential $V_A(r)$.

The completeness of the set of energy eigenfunctions allows us to have eigenfunction expansion of the energy GF as

$$G_A(r, r_0; \epsilon_n^A) = \sum_{n=0}^{\infty} \frac{\varphi_A(r)\varphi_A^*(r_0)}{E - \epsilon_n^A - i\varepsilon} \quad (3)$$

from which we read off the analytic form of the wave function of the solved quantum system.

The ET includes a coordinate transformation, which is followed by a functional transformation and a set of plausible ansatze to restore the transformed equation to standard Schrodinger GF equation form.

We now invoke the coordinate transformation

$$r \to g_B(r), \qquad r_0 \to g_B(r_0) \quad (4)$$

which is followed by a functional transformation

$$G_B(r, r_0; \epsilon_n^B) = f_B^{-1}(r) G_A(g_B(r), g_B(r_0), \epsilon_n^A) f_B^{-1}(r_0) \quad (5)$$

that the resulting equation is of the same form as (1) but with new parameters and leads to the following equation

$$\left[\partial_r^2 + \left(\frac{d}{dr} ln \frac{f_B^2(r) g_B^{D_A - 1}(r)}{g'_B(r)}\right)\partial_r + \left(\frac{d}{dr} ln f_B(r)\right)\left(\frac{d}{dr} ln \frac{f'_B g_B^{D_A - 1}(r)}{g'_B(r)}\right) + g'^2_B(\epsilon_n^A - V_A(g_B))\right] G_B(r, r_0; \epsilon_n^A) = g'^2_B f_B^{-1}(r) \frac{\delta\left(g_B(r) - g_B(r_0)\right)}{g_B^{D_A - 1}(r_0)} f_B^{-1}(r_0) \quad (6)$$

The transformation functions $g_B(r)$ and $g_B(r_0)$ are smooth differentiable function, which are at least three times differentiable function and $f_B^{-1}(r)$, $f_B^{-1}(r_0)$ are the modulating function required to mould the above equation to the standard Schrodinger GF equation form. We make the coefficient of the first order derivative equal to $\frac{D_B - 1}{r}$, fixing the functional form of

$$f_B^{-1}(r) = C_N g'^{\frac{1}{2}}_B(r) g_B^{\frac{D_A - 1}{2}}(r) r^{-\frac{D_B - 1}{2}}(r) \quad (7)$$

where C_N is the normalization constant.

therefore equation (5) and (6) lead to

$$G_B(r, r_0; \epsilon_n^B) = g'^{\frac{1}{2}}_B(r) g_B^{\frac{D_A - 1}{2}}(r) r^{-\frac{D_B - 1}{2}} \times G_A(g_B(r), g_B(r_0); \epsilon_n^A) g'^{\frac{1}{2}}_B(r_0) g_B^{\frac{D_A - 1}{2}}(r_0) r_0^{-\frac{D_B - 1}{2}}(r_0) \quad (8)$$

The right hand side of equation (6) can be simplified to $\frac{\delta(r - r_0)}{r_0^{D_B - 1}}$

which changes equation (6) to

$$\left[\partial_r^2 + \frac{D_B - 1}{r}\partial_r + \frac{1}{2}\frac{g'''_B}{g'_B} - \frac{3}{4}\left(\frac{g''_B}{g'_B}\right)^2 - \frac{D_A - 1}{2}\frac{D_A - 3}{2}\left(\frac{g'_B}{g_B}\right)^2 + \frac{D_B - 1}{2}\frac{D_B - 3}{2}\frac{1}{r^2} + g'^2_B\left(\epsilon_n^A - V_A\left(g_B(r)\right)\right)\right] \times G_B(r, r_0; \epsilon_n^B) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}} \quad (9)$$

In case of multi-term A-QS, we have to select a term of $V_A\left(g_B(r)\right)$ as working potential (WP) to implement ET and is designated as $V_A^W\left(g_B(r)\right)$.

In order mould equation (9) to the standard form of the Schrodinger GF equation, the following plausible ansatze have to be made, which are integral part of the transformation method.

$$g'^2_B V_A^W\left(g_B(r)\right) = -\epsilon_n^B \quad (10)$$

$$V_B^{(1)}(r) = -g'^2_B \epsilon_n^A \quad (11)$$

$$V_B^{(2)}(r) = g'^2_B\left(V_A\left(g_B(r)\right) - V_A^W\left(g_B(r)\right)\right) \quad (12)$$

$$V_B^{(3)}(r) = -\frac{1}{2}\frac{g_B'''}{g_B'} + \frac{3}{4}\left(\frac{g_B''}{g_B'}\right)^2 + \frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g_B'}{g_B}\right)^2 - \frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2} \qquad (13)$$

We obtain the new potential $V_B(r)$ as

$$V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r) \qquad (14)$$

The final form of the radial Schrodinger GF equation for B-QS established in an Euclidean space of the chosen dimension D_B is:

$$\left[\partial_r^2 + \frac{D_B-1}{r}\partial_r + \epsilon_n^B - V_B(r)\right] G_B(r, r_0; \epsilon_n^B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}} \qquad (15)$$

From equation (4), (5) and (8), the eigenfunction expansion of B-QS Green's function is

$$G_B(r, r_0; \epsilon_n^B) = \sum_{n=0}^{\infty} \frac{\left(\frac{g_B^{D_A-1}(r)}{g_B'(r) r^{D_B-1}}\right)^{\frac{1}{2}} \varphi_A(g_B(r))\varphi_A^*(g_B(r_0)) \left(\frac{g_B^{D_A-1}(r_0)}{g_B'(r_0) r_0^{D_B-1}}\right)^{\frac{1}{2}}}{E - \epsilon_n^B - i\varepsilon} \qquad (16)$$

The B-QS eigenfunctions $\varphi_B(r)$ can be read off from equation (16)

$$\varphi_B(r) = C_N \left(\frac{g_B^{D_A-1}(r)}{g_B'(r) r^{D_B-1}}\right)^{\frac{1}{2}} \varphi_A(g_B(r)) \quad (17)$$

and is known, since $\varphi_A(r)$ and $g_B(r)$ are known.

The normalization constant C_N is evaluated by using the following normalization condition for $\varphi_B(r)$ as

$$\int_0^\infty \varphi_B^2(r) r^{D_B-1} dr = |C_N|^{-2} = finite \qquad (18)$$

The normalization constant is given by

$$C_N = \left[\frac{-\epsilon_n^B}{\langle V_A^W(g_B(r))\rangle}\right]^{1/2}$$

The expectation value of ESP is always finite and so a part of it is also finite.

3.CONSTRUCTION OF EXACTLY SOLVABLE POTENTIAL FROM TRIGONOMETRIC SCARF POTENTIAL

3.1. First-order transformation

We consider Trigonometric Scarf QS [19] as a typical representative of a QS with non-power law potential henceforth called A-QS in a D_A-dimensional Euclidean space, which admits only S-wave ($l = 0$) eigenfunctions. The potential is

$$V_A(r) = (\mu^2 + \lambda^2 - \mu\alpha) sec^2 \alpha r - \lambda(2\mu - \alpha)\tan\alpha r \sec\alpha r \qquad (19)$$

which satisfies the GF equation in 1-dimensional space

$$\left[\partial_r^2 + E_n^t - \left((\mu^2 + \lambda^2 - \mu\alpha) sec^2\alpha r - \lambda(2\mu - \alpha)\tan\alpha r \sec\alpha r\right)\right] G_t(r, r_0; E_n^t) = \frac{\delta(r-r_0)}{r_0^{D_A-1}} \qquad (20)$$

Energy eigenvalues are

$$E_n^t = (\mu + \alpha n)^2 \qquad (21)$$

BOUND STATE SOLUTION

The eigenfunctions expansion in terms of GF for bound state $((\mu + \alpha n)^2 < 0)$ with trigonometric scarf potential

$$G_t(r, r_0; E_n^t) = \sum_{n=0}^{\infty} \frac{\varphi_t^{(n)}(r)\varphi_t^{*(n)}(r_0)}{E - (\mu + \alpha n)^2 - i\in} \qquad (22)$$

The exact eigenfunctions in terms of Jacobi polynomial $P_n^{(\beta,\gamma)}(\sin\alpha r)$ as

$$\varphi_t^{(n)}(r) = N_t(1 - \sin\alpha r)^{\frac{\mu-\lambda}{2\alpha}}(1 + \sin\alpha r)^{\frac{\mu+\lambda}{2\alpha}} P_n^{(\beta,\gamma)}(\sin\alpha r) \qquad (23)$$

where $= \frac{\mu}{\alpha} - \frac{\lambda}{\alpha} - \frac{1}{2}$, $\gamma = -\frac{\mu}{\alpha} - \frac{\lambda}{\alpha} - \frac{1}{2}$ and $n = 0,1,2, \ldots \ldots < \mu$, N_t-normalization constant.

The trigonometric scarf QS can be transformed by ET to hyperbolic scarf QS, the potential is denoted by $V_h(r)$. The generated new QS in D_B-dimensional space is

$$\Big[\partial_r^2+\frac{D_B-1}{r}\partial_r+g'^2_h(E_n^t-(\mu^2+\lambda^2-\mu\alpha)sec^2\alpha g_h(r)-\lambda(2\mu-\alpha)\tan\alpha g_h(r)\sec\alpha g_h(r))+\frac{1}{2}\frac{g_B'''}{g_B'}-\frac{3}{4}\left(\frac{g_B''}{g_B'}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\Big]G_h(r,r_0;E_n^h)=\frac{\delta(r-r_0)}{r_0^{D_B-1}} \quad (24)$$

The relation between GFs $G_t(r,r_0;E_n^t)$ and $G_h(r,r_0;E_n^h)$ is

$$G_h(r,r_0;E_n^h)=f_B^{-1}(r)G_t(g_h(r),g_h(r_0);E_n^t)f_B^{-1}(r_0) \quad (25)$$

and the eigenfunctions in terms of GF $G_h(r,r_0;E_n^h)$ are

$$G_h(r,r_0;E_n^h)=\sum_{n=0}^{\infty}\frac{r^{-\left(\frac{D_B-1}{2}\right)}g_h'^{-\frac{1}{2}}(r)\varphi_t^{(n)}(g_h(r))\varphi_t^{(n)}(g_h(r_0))r_0^{-\left(\frac{D_B-1}{2}\right)}g_h'^{-\frac{1}{2}}(r_0)}{E-E_n^h-i\in} \quad (26)$$

and, it is completely specified once transformation function $g_h(r)$ is known.

From equation (26) we can evaluate energy eigenfunctions

$$\varphi_h^{(n)}(r)=r^{-\left(\frac{D_B-1}{2}\right)}g_h'^{-\frac{1}{2}}(r)\varphi_t^{(n)}(g_h(r)) \quad (27)$$

The trigonometric Scarf potential is a two term potential (equation 19) and we have three (2^2-1) choices to select the WP. If one chooses $(\mu^2+\lambda^2-\mu\alpha)sec^2\alpha r$ as the WP. Ansatze (10)-(13) , require to bring equation (24) into the standard Schrodinger GF equation form, are now

$$g'^2_h\big((\mu^2+\lambda^2-\mu\alpha)sec^2\alpha g_h(r)\big)=-E_n^h \quad (28)$$

Integration of equation (28) yields

$$g_h(r)=\frac{1}{\alpha}\sin^{-1}(\tan\alpha\xi_n r) \quad (29)$$

where

$$\xi_n=\left(-\frac{E_n^h}{\mu^2+\lambda^2-\mu\alpha}\right)^{1/2} \quad (30)$$

The integration constant is put equal to zero which attributes the local property $g_h(0)=0$.

Now equation (11) and (29) lead to

$$V_h^{(1)}(r)=-\xi_n^2E_n^t sech^2\alpha\xi_n r=-C_h^2 sech^2\alpha\xi_n r \quad (31)$$

where C_h^2 is the characteristic constant 0f B-QS and is

$$C_h^2=\xi_n^2E_n^t \quad (32)$$

which will give the energy eigenvalues of B-QS.

$$V_h^{(2)}(r)=g'^2_h(V_t(g_h(r))-(\mu^2+\lambda^2-\mu\alpha)sec^2\alpha g_h(r)) \quad (33)$$

as $D_A=1$

$$V_h^{(3)}(r)=\frac{1}{2}\frac{g_B'''}{g_B'}-\frac{3}{4}\left(\frac{g_B''}{g_B'}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2} \quad (34)$$

The multi-term B-QS potential $\left(V_h(r)=V_h^{(1)}(r)+V_h^{(2)}(r)+V_h^{(3)}(r)\right)$ now becomes:

$$V_h(r)=\left(-C_h^2+\frac{1}{4}\rho_n^2\right)sech^2\rho_n r-\sigma_n tanh\rho_n r+\frac{1}{4}\rho_n^2tanh^2\rho_n r+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2} \quad (35)$$

Where $\rho_n=\alpha\xi_n$ and $\sigma_n=\mu(2\mu-\alpha)\xi_n^2$. Corresponding energy spectrum from equation (32) is

$$E_n^h=-\frac{C_h^2(\mu^2+\lambda^2-\mu\alpha)}{(\mu+\alpha n)^2} \quad (36)$$

which is a Sturmian QS, it cannot be made normal/physical, because in the presence ρ_n and σ_n it is found that, by explicit calculation that both of them cannot be made n-independent simultaneously in a consistent manner. However its "desendent" QS can be reached through ET may become normal.

Invoking the equations (35) and (36) on equation (24), the radial Schrodinger GF equation for B-QS takes the form:

$$\Big[\partial_r^2+\frac{D_B-1}{r}\partial_r+g'^2_h\left(E_n^h-\left(-C_h^2+\frac{1}{4}\rho_n^2\right)sech^2\rho_n r-\sigma_n tanh\rho_n r+\frac{1}{4}\rho_n^2tanh^2\rho_n r\right)+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\Big]G_h(r,r_0;E_n^h)=\frac{\delta(r-r_0)}{r_0^{D_B-1}} \quad (37)$$

The eigenfunction expansion in terms of GF of B-QS is given by equation (26), from where we read off the exact eigenfunctions $\varphi_h(r)$ as in equation (27) in D_B-dimensional takes the form:

$$\varphi_h(r) = N_h r^{-\left(\frac{D_B-1}{2}\right)} cosh^{1/2}\rho_n r(1-tanh\rho_n r)^{\frac{(\mu-\lambda)\xi}{2\rho_n}}(1+tanh\rho_n r)^{\frac{(\mu+\lambda)\xi}{2\rho_n}} P_n^{(\beta,\gamma)}(tanh\rho_n r) \quad (38)$$

The normalizability of $\varphi_h(r)$ obtain bt ET can be proved under general condition. Normalizability condition for D_B-dimensional B-QS eigenfunctions is

$$\int_0^\infty \varphi_h^2 r^{D_B-1}\,dr = \frac{1}{|N_h|^2} = finite \quad (39)$$

From equation $V_h^{(1)}(r) = g'^2_h E_n^t$ (equation 31) it is reduce to

$$|N_h|^2 \frac{\langle V_t^{(W)}(r)\rangle}{\frac{C_h^2(\mu^2+\lambda^2-\mu\alpha)}{(\mu+\alpha n)^2}} = 1$$

where $\langle V_t^{(W)}(r)\rangle$ is the expectation value of $V_t^{(W)}(r)$ w.r.t. A-QS eigenfunctions.

We can derive Ramanovski polynomial by means of following substitutions:

$\mu \to i\mu$, $\frac{\rho}{\xi} \to i\frac{\rho}{\xi}$, $\frac{\mu\xi}{\rho} \to -\frac{\mu\xi}{\rho}$ and $\lambda \to \lambda$

with these substitutions equation (38) becomes

$$\varphi_h(r) = N_h r^{-\left(\frac{D_B-1}{2}\right)} cosh^{1/2}\rho_n r(1+itanh\rho_n r)^{\frac{(\mu+i\lambda)\xi}{2\rho_n}}(1-itanh\rho_n r)^{\frac{(\mu-i\lambda)\xi}{2\rho_n}} P_n^{(\beta,\gamma)}(-itanh\rho_n r) \quad (40)$$

Corresponding Jacobi polynomial equation is

$$(1+tanh^2\rho_n r)P_n''^{(\beta',\gamma')}(i\,tanh\rho_n r) + \left\{\frac{2\beta i}{\alpha} - \left(1-\frac{2\mu}{\alpha}\right) itanh\rho_n r\right\} P_n'^{(\beta',\gamma')}(i\,tanh\rho_n r) + n\left(n-\frac{2\mu}{\alpha}\right) P_n^{(\beta',\gamma')}(i\,tanh\rho_n r) = 0 \quad (41)$$

where $\beta' = -\frac{\mu\xi+i\lambda\rho}{\rho_n} - \frac{1}{2}$ and $\gamma' == -\frac{\mu\xi-i\lambda\rho}{\rho_n} - \frac{1}{2}$

The differential equation satisfied by the complex Jacobi polynomial

$$(1+tanh^2\rho_n r)R_n''^{(p,q)}(\tanh\rho_n r) + \{2(-p+1)\tanh\rho_n r\}R_n'^{(p,q)}(\tanh\rho_n r) - \{n(n-1)+2n(1-p)\}R_n^{p,q}(tanh\rho_n r) = 0 \quad (42)$$

Equation (41) and (42) are the identical equation and differ by a phase factor i^n. comparing them by $\beta' = -p - \frac{iq}{2}$ and $\gamma' = \beta^*$.

As a real polynomial (non-classical polynomial) exists, popularly known as Ramanovski polynomial, which is obtainable from Jacobi polynomial. The Ramanovski polynomials are related to the complex Jacobi polynomials via

$$R_n^{p,q}(\tanh\rho r) = i^n P_n^{\left(-p-\frac{iq}{2},-p+\frac{iq}{2}\right)}(i\tan\rho r) \quad (43)$$

SCATTERING STATE SOLUTION

We are concern with continuous part of the energy spectrum, which is corresponds to the positive energy of the GF equation and these eigenfunctions are of unbound states. The potential energy $V_h(r)$ decreases in magnitude as the $r = |r|$ from the scattering centre, become large, such that $\lim_{r\to\infty} V_h(r) \equiv 0$. Total energy of the particle is therefore $E_n^h = \frac{\hbar}{2m}k^2 = k^2$ (in atomic units). It is the corresponding wave number. In scattering state $E_n^h = k^2 > 0$. so we can replace E_n^h by $k^2$and write the analogue of equation (26) as

$$G_h(r,r_0;k^2) = \int_0^\infty \frac{dE}{E-k^2-i\in} r^{-\left(\frac{D_B-1}{2}\right)} cosh^{1/2}k'r(1+itanhk'r)^{\frac{(\mu+\lambda i)\xi}{2\rho_n}}(1-itanhk'r)^{\frac{(\mu-\lambda i)\xi}{2\rho_n}} P_n^{(\beta',\gamma')}(itanhk'r) r_0^{-\left(\frac{D_B-1}{2}\right)}(1+itanhk'r_0)^{\frac{(\mu+\lambda i)\xi}{2\rho_n}}(1-itanhk'r_0)^{\frac{(\mu-\lambda i)\xi}{2\rho_n}} P_n^{(\beta',\gamma')}(itanhk'r_0) \quad (44)$$

where $k' = \frac{k}{\sqrt{\mu^2+\lambda^2-\frac{\mu\rho}{\xi}}}$

Imaginary part alone gives the continuous energy eigenvalues and hence

$$ImG_h(r,r_0;k^2) = \pi \int_0^\infty dE\delta(E - k^2) r^{-\left(\frac{D_B-1}{2}\right)} cosh^{\frac{1}{2}} k'r (1 + itanhk'r)^{\frac{(\mu+\lambda i)\xi}{2\rho_n}} (1 - itanhk'r)^{\frac{(\mu-\lambda i)\xi}{2\rho_n}} P_n^{(\beta',\gamma')}(itanhk'r) r_0^{-\left(\frac{D_B-1}{2}\right)} (1 + itanhk'r_0)^{\frac{(\mu+\lambda i)\xi}{2\rho_n}} (1 - itanhk'r_0)^{\frac{(\mu-\lambda i)\xi}{2\rho_n}} P_n^{(\beta',\gamma')}(itanhk'r_0) \quad (45)$$

The scattering wave $\varphi_h^{scatt}(r,k^2)$ is

$$\varphi_h^{scatt}(r,k^2) = \pi \int_0^\infty r^{-\left(\frac{D_B-1}{2}\right)} cosh^{1/2} k'r (1 + itanhk'r)^{\frac{(\mu+\lambda i)\xi}{2\rho_n}} (1 - itanhk'r)^{\frac{(\mu-\lambda i)\xi}{2\rho_n}} P_n^{(\beta',\gamma')}(itanhk'r) \quad (46)$$

3.2 Second order transformation

Application of ET on the B-QS comprising equation (35) and (37) we can generate another QS (say C-QS). The choice of WP from the term which is directly comes from the energy term of the parent QS, revert it back to the parent (Trigonometric Scarf) QS. In this particular case $\left(-C_h^2 + \frac{1}{4}\rho_n^2\right) sech^2 \rho_n r$ is directly coming from the energy term of Trigonometric Scarf QS. When we take $-\sigma_n tanh\rho_n r$ as the WP the ansatze (10) now becomes $g'^2_C(-\sigma_n tanh\rho_n g_C(r)) = -E_n^C$. The integration is$\int \sqrt{-tanh\rho_n g_C(r) dg_C} = \sqrt{\frac{E_n^C}{\rho_n}} + C$, however the functional dependence cannot be inverted to get an analytic expression for $g_C(r)$.

4. DISCUSSION AND CONCLUSION

We have obtained exactly solvable potential of radial Schrodinger GF equation in D-dimensional Euclidean space whose bound state as well as scattering state solutions are given for hyperbolic Scarf of QSs. We have use a simple and compact mapping procedure called ET method, which consist of co-ordinate transformation followed by a functional transformation (FT). The parent system is in one dimensional (Euclidean) space, but FT component of ET allows a consistent way to choose the dimension of the transformed system other than two dimensions. The transformation method is performed on the linear second order differential equation satisfied by a particular special function to retrieve radial Schrodinger GF equation. The constructed potentials are non-power law type with a background inverse square potential $(D-1)(D-3)r^{-2}$ which vanishes for $D = 1$ and $D = 3$.

The hyperbolic Scarf potential has various applications in physics. In solid state physics it is used in the construction of more realistic periodic potential in crystal. In electrodynamics hyperbolic Scarf potential appears in a class of problems with non-central potential. In particle physics it has fine application in studies of the non-perturbative sector of gauge theories by means of toy models such as the scalar field theory in space-time dimensions.

In order to get the scattering state of hyperbolic Scarf potential we replace E_n^h by k^2 and write the analogue of equation (26). To determine the wave function belonging to continuum we use the symbolic identity $\lim_{r\to 0} \frac{1}{(E-k^2)\pm i\epsilon} = P\frac{1}{E-k^2} \mp i\pi\delta(E - k^2)$ which yields the expression for the scattering wave as equation (46).

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