Construction and Working of Numbers by Peano /Dedekind /Peirce Axioms with Peano Arithmetic

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Abstract
The symbol N is usually reserved for the set of Natural numbers. Throughout this paper, we present all familiar properties of Natural numbers including basic Arithmetic operations on N given by Peano, Dedekind and Peirce. The basic study will be to see how the construction and working of Natural numbers is possible in mathematics and what is the history behind that. Peano axioms are characterized by version of Number Theory known as Peano Arithmetic Mathematical Induction in Discrete Theory and how to obtain ring of integers from five axioms of Peano including properties of integers.
***Keywords*:** Natural numbers N, Set of Integers, Ring$R$, Binary Operation, Successor function, Peano Arithmetic.

INTRODUCTION:

The basic of Mathematics is Set Theory. In any Mathematical system, there are undefined terms which are related to one another by some logical statements called postulates.

Richard Dedekind (1831-1916) published in 1888 a paper entitled "**Was sind and was sollen die Zahlen**?" translated by various as "**What are numbers and What should they be?"**

Before talking about Peano, Dedekind's Axioms, we must know about basic terms like set, subsets and its notation, empty Set, Functioning, Mappings (Operations or Correspondence), relation between two sets, Binary operations etc. If X is a non-empty set and let any element$'a'$belongs to$X$ then it is written as$'a \in X'$, if it is not then $a \notin$X. Natural numbers, Integers, Rational Numbers, Real numbers and Complex Numbers are denoted by the symbol N, Z, Q, R, C respectively. In this paper, we will be only familiar with Natural Numbers and Integers.$N = \{0,1,2,3,4 \ldots\}$ which is basically known as Natural set.

$\{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$ is the set of Integers which is denoted by Z where Z is taken from the GERMAN word "**ZAHLEN**" which stands for numbers.

A set always contain well defined and different objects. Sets are always denoted by capital letters and its elements are denoted by small letters. $Y \subseteq X$means X is super set of Y and Y is subset of X which shows that all elements of Y in X.Also a function f from X to Y is denoted by $X \rightarrow Y$ where X and Y are called domain and co-domain respectively and its image is denoted by $f(a)$where$a \in X$ and $f(a) \in Y$. When we defined an operation on function, then its working as mappings like one–one, Many one, Into, Onto, One One into, One One Onto, Many One into, Many one onto, Identity, Constant, Equal etc. which takes input to get output. A Binary relation on a set X is said to be equivalence and write a~b if it is Reflexive $a \sim a \; \forall \, a \in X$, Symmetric $a \sim b \Rightarrow b \sim a \; \forall \, a, b \in X$ Transitive if $a \sim b, b \sim c \Rightarrow a \sim c \; \forall \, a, b, c \in X$ after that we can easily establish all the familiar properties of Natural numbers (i.e., positive integers).

Peano Axioms: In 19th century Italian Mathematician "Giuseppe Peano" presented the Peano's axioms which is also known as Dedekind Peano Axioms or the Peano postulates. These axioms are for the natural numbers. In 1860, Hermann Grassmann showed that many facts in arithmetic could be derived from more basic facts about successor operations and induction. In 1881, Charles Sanders Peirce provided an axiomatization of natural numbers. In 1888 Richard Dedekind proposed another axiomatization of natural numbers system and in 1889 Peano published a simplified version of all of them as a collection of axioms in his book ***The Principles of Arithmetic*** presented by a new method in ***LATIN*** written as ***Arithmetics Principia, nava Methodo exposita.***

The five axioms are given by Peano may be read as

I. $1 \in N$ i.e., one is a natural number.

II. For each $x \in N$ there exist a unique $x' \in N$ is called the successor of x or we can say that any natural number produces another natural number by taking sign "+" to the number xi.e.,∃ a map $x \to x'$ of N into itself which is called successor map.

III. $x^{'} \neq 1$ for $x \in N$i.e., one is not successor of any natural number.

IV. $\forall x, y \in N\,,\ x^{'} = y^{'} \Rightarrow x = y$. In other words, we sayif x and y are two natural numbers and their successor are equal, then they are also equal.Mathematically we can also say that, the successor map $x \to x^{'}$ of N intoN is injective map.

V. Let A be a subset of N such that $x \in A$ then $x' \in A$ then $A = N$ that means we say if A is a class which contains one, and if the class made up of the successor of A is also contained in A, then every member is contained in the class A and then A become to be N.

Some facts which are not included in Peano axioms are written as follow

1. Zero is a natural number.

But the Peano's axioms does not say anything about the zero, these axioms start with saying that one is natural number as we know that the first natural number is 1 so we can say the choice in axiom first is arbitrary and it is chosen for some reason because zero is working as additive identity in arithmetic and it plays very important role to make a ring of integers.

The fifth axiom is become to be axiom of induction or the first principal of induction because we have 1'(1+1)=2 , 2'(2+1)=3,3'(3+1)=4 and so on…

The first principal of induction can also be studied in a following way:

Let us consider for every natural number $n \in N$ we have a statement written as $T(n)$ such that

- $T(n)$ is true statement
- T(n) is true implies $T(n+1)$ is also true that means $T(n)$ is true for all $n \in N$.

(According to Richard Dedekind)

The nature and meaning of Numbers can be defined as "Numbers are free creations of human mind"; these numbers help human to understand the meaning of things and their difference very easily. Its only through the purely logic process of building up the science of numbers and by thus acquiring the continuous number domain that we prepare accurately investigate our notion of space and time by bringing them into relation with this number domain created in our mind. If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible.

How Dedekind worked on axioms of Natural numbers

- As Dedekind think that anything in our life can be easily taken as a numberand anything is same as to another thing or it is also not so it depend on the nature of that thing sometimes it is working like that. Suppose in Mathematically or in symbolically if we have two things like $A \text{ and } B$ if $A = B$ then obviously we can say $B = A$ and then $A = A$ always. Next if we have three things naming as $A, B \text{ and } C$ and $A = B$, $B = C$ then $A = C$ and if A is not same as B then we can say that$A$ and B have not relation or not belong to each other.

- In real life, we can see different types of things just like table, chair, carpet, bed, water, salt, motorcycle, vegetables etc. These all things are in systematic and these are all daily uses things and we can see these types of things in every house $say\ H$. We can easily say that all these things are of Hmeans they contain in it. Similarly different numbers are also contained in some other type of system i.e., in A(consider).If we find same type of system as like H belong to then we say that both systems belongs same things in symbols like $H = A$. Sometimes a particular thing like car don't belong to H so we say car is not the element in H.
- $\emptyset$ element (not all) are part of B system means a set. In this case, Dedekind says H is whole of B.
- Transformation of a system:By transformation, he simply wants to say that transformation $\emptyset$ of a system S we understand that every element in a system can be easily transformed i.e. become $\emptyset(s)$.
  For our ease if we take an example that if there are pieces of wood in a shop can be transformed to chair, table, window after applying some certain rule.
  The simplest transformation of a system that if elements transformed into itself.
- Similar Systems**:** A transformation $\emptyset$ is distinct or similar when two different elements $a \text{ and } b$ of system S there correspond different transformations i.e. $a' = \emptyset(a)$ and $b' = \emptyset(b)$. But if $a' = b' i.e. \emptyset(a) = \emptyset(b)$ then it is also possible that $a = b$.
- We find a chain of system for which if Q a chain if we find a system Q' which is whole in Qi.e., when $Q' \subseteq Q$
- A finite and infinite system: A system is said to be infinite when its similar to some infinite part of similar transformation otherwise finite.
- Series of Natural Numbers**:** The system N is simply infinite when $\exists$ a similar transformation $\emptyset\ from\ N$ into N itself. According to Dedekind, Symbol '1' is the base element of N and N is simply infinite system which is set in order by transformation $\emptyset$. If we retain the earlier convenient symbol for transformations and chains then we find N is infinite system and an element 1 which is satisfied the following conditions l,m,n,o

$$\boldsymbol{l \to N' \subseteq N}$$
$$\boldsymbol{m \to N = 1}$$
$$\mathbf{n \to The\ element\ 1\ is\ not\ contained\ in\ N'}$$
$$\mathbf{O \to The\ transformation\ \emptyset\ is\ similar}$$

He gives above four axioms for natural numbers. According to him,1 is taken as base of Natural number series N. Also used complete induction so that above axioms are hold for all numbers n of a chain so its only sufficient to prove for any finite no $m = n$.

C.S Peirce on the logic of numbers:

On dated 1881-01-01 Charles Sanders Peirce published a paper on the logic of numbers in American Journal of Mathematics. It is an article from volume 4 and its free available on JSTOR. In this paper, he (C.S.Peirce) defines the transitive relation between three number of quantities. According to him, a system which is simple contain either "as great as" or "as small as" quantities to all others. A simple system of Quantity is ether continuous, discrete or continuous/discrete mixed. We define these as follows:

- A continuous system: In which every quantity greater than another is also greater some intermediate quantity greater than that other i.e. let three quantities areq, l, r $then$ $q > l > r$
- A discrete system: In which every quantity greater than another is next greater than some quantity(that is, greater than without being greater than something greater than) i.e., let a, b, c, d, e, f are $quantities$ $then$ $a > d, b > a, c > f, d > e, e > c, f > b$. all these are greater than some other quantity.
- A mixed system: In which some quantities greater than others are next greater than some quantities while some are continuously greater than some quantities. i.e.,let a, b, c, d, e, f are $quantities$ $then$ $a > b, a > c, a > f$ $, e > b, e > f$

Discrete quantity is further of three types that are limited, semi-limited or unlimited. We can define these system as follows:
A limited system has an absolute maximum and an absolute minimum quantity.
A semi limited system has only (generally considered a minimum) without the other ;an unlimited system has neither maximum nor minimum.
Clauses: -

1. The minimum number is one. Take $a + b$ If $a = 1$, the number next greater than b means a+b=1+b. In other words the next greater than number is $a' = 1 + b$
2. $a \times b$(the closure property under multiplication) is meant and in other cases $b + a'b$ $i.e.$ $b(1 + a')$ where $a'$is the number next smaller than a as above we consider in clause 1.

In order to prove some results, we use want to make true above clauses

1. Associative principle of addition:
   $(a + b) + c = a + (b + c)$where a, b and c are any natural numbers. By induction we see that First it true for $a = 1$ so $(1 + b) + c = 1 + (b + c)$ and 2nd step is true for $x = n$, it true for $x = 1 + n$ i.e., if $(n + y) + z = n + (y + z)$ then $((1 + n) + y) + z = (1 + n) + (y + z)$
2. To prove commutative principle of addition that $x + y = y + x$
   If $x = 1, y = 1$ then it is identity.
   If $x = n, y = 1$ then it is also true for $x = 1 + n, y = 1$i.e.,$n + 1 = n + 1$ than $(1 + n) + 1 = 1 + (1 + n)$ so we can also see that $x + (1 + n) = (1 + n) + x$
3. If '.' Denote the operation of multiplication then
4. The distributive principle now first clause then by second, we check for $x = 1, x = n$ and $x = 1 + n$
   1st is $(x + y).z = x.z + y.z$ where x,y,z are any natural numbers .
   2nd is $x.(y + z) = x.y + x.z$
5. Commutative principle of multiplication is that $x.y = y.x$ where x and y are any natural number.
6. There is another small result $x + y = x + z$ then $y = z$ we check result for $x = 1, x = n$ and also true for $x = 1 + n$ it becomes $1 + n + y = 1 + n + z$
7. It also works on counting of numbers whether number of elements are finite or infinite. so, there are various types of properties which are found in this paper of Natural numbers.

SURVEY OF LITERATURE:

We are present some Mathematicians Richard Dedekind (1888) and Giuseppe Peano(1889),Booklets on the foundation of Arithmetic Working on numbers i.e. Peano Dedekind axioms for Natural numbers *About Dedekind :*

In December 1887, sind und was sollen die zahlen was his first publicationBraunschiveig: Friedrich Vieweg&sons,1888. 54 pages that was issued in December 1887.

In 1872-1878, his first book draft was written but his Main Manuscript was destroyed, buthis first book draft was written in 1872-1878 preserved, Cod Ms Dedekind, III, 1 transcribed in [Dugac,1976,293-309]. Also, fragments of a second draft of 1887 [ibidem, III, 1, III].After that there was a 2nd editioncame in 1893 and then3rdeditioncame in 1911.Both Vieweg[Various photoreprintsuntilthe so called '10th edition came in 1969,prefaced by G, Asses(itself represent) .Also in *GesammeltemathematischeWerke, Vol 3,Braunschwerg;Vieweg 1932,335-391]*

After that he translated in English under the title 'The Nature and Meaning of Numbers' (trans. W.W.Beman), in R.Dedekind, Essays on the theory of numbers, Chicago, open court ,1901,29-115.[several reprs.inc.New York:Dover,1963]Further It is Revised by W.Ewald in his edition 'From Kant to Hilbert'New York: Oxford University Press,1996(790-833.2) 'what are numbers and what should they be?' (ed. and trans. H.Pogorzelski, W.Ryan and W.Synder)Orono, Maine: Research Institute for Mathematics,1995'.

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The first publication was Arithmetics Principia, Turrin: Bocca published in 1889of16+20 Pages.

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The full work given by Guiseppe Peano is translated by H.C. Kennedy in his edition at London:George Allen and Unwin,1973,101-134.There are various articles related to this given by mathematicians Grassmann, Drichlet, Dedekind on irrational numbers, Boole, Riemann on Trignometric Series, Cantor, Whitehead and Russell. Now we will discuss the Dedekind, Peano and Peirce theory with respect to Natural Numbers. In this we mainly discussed the contents of books written by above mentioned authors. He realized that a general theory of sets and mappings is a sufficient foundation for the natural numbers and indeed the usual set theoretic definitions of N are the exemplify his ideas.

Now the following are some of the immediate consequences of Piano's axioms.

1. For any $a \in N, a' \neq a$.
2. For any $a \in N, a \neq 1$, there exist a unique $b \in N$ such that $a = b'$.
3. There exist only one and one binary operation" + " in satisfying(i) $a + 1 = a'$(ii) $a + b' = (a + b)'$ for all $a, b \in N$
4. The binary operation + in N satisfies the following laws:(i)for all $a, b, c \in N, (a + b) + c = a + (b + c)$ (associative law of addition) (ii)for all $a, b \in N, (a + b) = (b + a)$ (commutative law of addition)
5. Let $a, b \in N$.Then one of the following statements holds(I) $a = b$ (ii) $a = b +$
6. (iii) $b = a + v$ for some $u, v \in N$

7. Cancellation laws for addition (i)$a + u \neq a$ (ii) $a + x = a + y \Rightarrow x = y$
8. Trichotomy law of natural numbers: Given $a, b \in N$, one and only one of the following statements holds:(i)$a = b$ (ii) $a > b$ (iii) $a < b$
9. If $a \neq 1$, then $1 < a$
10. Let $a, b \in N$. then $a < b$ if and only if $a + 1 \leq b$
10 Well Ordering property of natural numbers: Every non empty set of natural numbers possesses a least member $i.e. T = \{n \in N \leq a\}$for all $a \in S$ has least element.
11 Second principle of induction: Let S be s subset of N such that $(i) 1 \in S$ (ii) $n \in S$ whenever $m \in S$ for all positive integers $m < n$. Then $S = N$
12 There exist one and only one binary operation "." in N satisfying $(i) a.1 = a (ii) a.b' = a.b + a$ for all $a, b \in N$
13 The binary operation of multiplication in N satisfies the following laws:$(i) a.b = b.a$ (commutative law) $(ii)(a.b).c = a.(b.c)$ (associative law) $(iii) a.(b + c) = a.b + a.c$ (distributive law)

INTEGERS:

After the study of the natural numbers now we go for the set of integers which is denoted by Z i.e. we proceed to a next system of numbers that are integers arises from the fact that an equation of the type $x = l + y$, that where $x, y \in N$, equation does not have a solution from the set of natural numbers.

We first check that the set $N \times N$ is satisfied an equivalence relation $'\sim'$ which is further form equivalence class said to be Z.

We canuse the binary operations under addition$' + '$and multiplication$'.'$under the following rules:

$$\overline{(x, y)} + \overline{(z, w)} = \overline{(x + z, y + w)}$$
$$(\overline{x, y})(\overline{z, w}) = (\overline{xz + yw, xw + yz})$$

so, we can easily prove this relation to reflexive, symmetric and transitive.

Now there are some theorems that based on Integers.

- $(Z, +, .)$is Integral domain with unity. Rule that is follow under addition is $(\overline{x, y}) + (\overline{y, x}) = (\overline{x + y, x + y}) = (\overline{1,1})$ and multiplication is$(\overline{x, y}).(\overline{z, w}) = (\overline{xz + yw, xw + yz}) = (\overline{1,1})$
- N embeds in Z under the mapping $m \to (m + 1,1), m \in N$ that preserves addition and multiplication.
- Trichotomy law of integers: under this if $a \in Z$, then one and only one hold of the following (i) $a = 0$ $(ii) a \in Z^+ (iii) - a \in Z^+$ equivalently, Z is the union of disjoints subsets $\{0\}$, Z^+ and Z^- where $Z^+$is collection of positive integers and $Z^-$is the collection negative integers.

NOTE: We have proved that the ring $(Z, +, .)$ contains a subset Z^+ that is (i) close under+, (ii)close under(iii) $a \in Z$,then one and only one is true:$a = 0$, $a \in Z^+, -a \in Z^+$. The set Z^+ is called set of positive elements of Z

Definition: Any ring R with any subset S of elements satisfying $(i), (ii) and$ (iii) in the above note is called an ordered domain.

Definition: Let $x, y \in Z$. if $x - y \in Z^+$ $thenx$ is said to be greater than yWe denote this by $x > y$

Definition: Let $x, y \in Z$. if $y - x \in Z^+$ $then$ x is said to be less than y We denote this by $y > x$

Here we see that a very interesting fact that Trichotomy law of integers also seen in other way that

i. $x = y$
ii. $x > y$
iii. $x < y$

- The order relations in Z satisfy the following laws
  i. Transitive law: If $x < y$ and $y < z$, then $x < z$
  ii. Addition to an inequality: If $x < y$ then $x + z = y + z$
  iii. Multiplication of an inequality: $x < y$ and $0 < z$, then $xz < yz$
  iv. Law of trichotomy: For any x and y in Z, one and only one of the relations $x = y, x > y$ or $x < y$ holds.
- In any ordered ring R (that satisfies the above result) all squares of non-zero elements are positive
- In any ordered ring R with unity I, we have $I > 0$.
- Any ordered domain D with unity whose positive elements are well Ordered is isomorphic to the ring of integers Z

CONCLUSION:

It is concluded that the set of Natural numbers are formed because of five axioms of Giuseppe Peano. There are lot of Mathematicians that worked on Natural numbers and gives improved axioms every time but there is some difference in Peano and other axioms given by mathematicians that Peano follow that Natural numbers starts from 1 wheather in present days we take first natural number as 0. In last we will say that Natural numbers also give the idea that how ring of integers formed and we find various important results related to integers.

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