Numerical solution of singular integral equation by using Modified Adomian Decomposition method &Homotopy Perturbation method

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Abstract: In this work, we are using two numerical approximation methods , Modified Adomian Decomposition method & Homogony Perturbation method to solve the singular integral equations by Laplace transformation method. Some numerical examples are given. A comparison is being done between exact solution and approximate solution. We are also computing the error of this method. All the calculations are done with help of MATLAB. A few examples including singular integral equation provided to show validity and applicability of this approaches.

Key Words: Modified Adomian Decomposition method, Homotopy Perturbation method, Volterra singular integral equation, Laplace transform method, Abel singular integral equation.

Subject classification: 44A10, 45E10.

Introduction: Applications of integral equations are being used in various fields, including Statistics, Mathematical Physics, Chemistry, heat Conduction, Fluid flow, Scattering theory, Chemical reaction, Population dynamics, Semi Conductors etc. Integral equation arises in the potential theory are more useful incomparison to other integral equations arises in other area of the research. Integral equation arises also in conformal mapping, diffraction problems, scattering in quantum mechanics, water waves, and population growth models [1-4]. Much attention has been devoted for the search of better and more efficient methods in recent years with the introduction of several numerical methods such as the Runge-Kutta methods, Galerkin methods, Chebyshev collocation methods, rationalized Haar functions method, Taylor collection methods, Galerkin methods with hybrid functions, and Adomian Decomposition Methods (ADM) [5-8]. The approximations obtained by the HAM are uniformly valid not only for small parameters. Until recently, the application of the homotopy analysis method in nonlinear problems has been devoted by scientists and engineers[9-11].

Singular Integral Equations: An integral equation of the form $y(x) = f(x) + \int_c k(x,t)y(t)dt$ is said to be singular if the range of integration is infinite, i.e., defined in the integral $0 < x < \infty$ or $-\infty < x < \infty$, or in which the kernel is discontinuous, i.e., if the kernel is not square integrable.

General Form of Abel Singular Integral Equation: The general form of Abel's integral equation is given by $f(x) = \int_a^x \frac{y(t)dt}{[h(x)-h(t)]^\alpha}$ $\forall\ 0 < \alpha < 1$.Where *h (t)* is strictly monotonically increasing and differentiable in [a, b] and $h^{'}(t) \neq 0$.

Weakly Singular Kernel: If the kernel of any integral equation is unbounded but the integral of the square of its modulus exists, then the equation may be solved directly by Fredholm's formulae. Such types of equations are known as weakly singular.

$y(x) = f(x) + \lambda\int_c \frac{H(x,t)}{|t-x|^\alpha} u(t)dt;$ $0 < x < 1$ is weakly singular. Here, *H(x, t)* is bounded function of two variables. Here, the kernel $k(x,t) = \frac{H(x,t)}{|t-x|^\alpha}$ is a weakly singular kernel.

Modified Adomian Decomposition Method for Abel integral equations:

We consider any functional equation of Abel integral equation by using Modified Adomian decomposition method.

$$y = f + Ny \tag{1}$$

Where f is a given function and N is a nonlinear operator. Modified Adomian Decomposition methods are assuming the solution in an infinite series for the unknown function *u(x).*

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (2)$$

Ny is a sum of decomposition series

$$Ny = \sum_{n=0}^{\infty} A_n \qquad (3)$$

Where $A_n^{'s}$ are the polynomials and $u_0, u_1, u_2, \dots \dots \dots u_n$ called the Modified Adomian polynomials and defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum \lambda^i y_i\right)\right]_{\lambda=0}, n \geq 0 \qquad (4)$$

In (1) substituting equation (2) and (3) then (1) become Abel's integral equation type, we get

$$\sum_{n=0}^{\infty} y_n(x) = f(x) + \sum_{n=0}^{\infty} A_n$$

If the component $y_0(x), y_1(x), y_2(x), \dots \dots \dots$ in used recurrence relation then we get

$$y_0(x) = f(x)$$

$$y_{n+1}(x) = A_n(y_0, y_1, y_2, \dots \dots \dots y_n), n \geq 0 \qquad (5)$$

But some authors have been proposed in modification ADM to Adomian Decomosition method. Modification method are evaluating in different type problem consider. We have been proposed in function f(x) is sum of the two functions $f_1(x)$ and $f_2(x)$ are suggested in following recurrence scheme.

$$y_0(x) = f_1(x)$$

$$y_1(x) = f_2(x)$$

$$y_{n+2}(x) = A_{n+1}(y_0, y_1, y_2, \dots \dots \dots y_n), n \geq 0$$

Modification ADM have been provided in more flexibility to the ADM and solving in complicated form integral equation.

Laplace MADM for Nonlinear Abel's Integral Equation:

Abel's integral equation are arises in two nonlinear forms; then first and second kinds as follows

$$y(x) = f(x) + \int_0^x \frac{F(y(t))}{\sqrt{x-t}}\,dt$$

Applying Laplace transform method in above equation on both sides, we get

$$y(s) = L\{f(x)\} + L\{k(x)\}L\{F(y(x))\} \qquad (6)$$

L is a Laplace transform and ADM have been defined in solution of u(x) an infinite series solution (2) then we get

$$y(s) = \sum_{n=0}^{\infty} y_n(s) \qquad (7)$$

If any nonlinear terms F(y(x)) are represented by Adomian Polynomials.

$$F(y(x)) = \sum_{n=0}^{\infty} A_n(x) \qquad (8)$$

Equation (2) and (8) substituting in equation (6) then we gets

$L\{\sum_{n=0}^{\infty} y_n(x)\} = L\{f(x)\} + L\{k(x)\}L\{\sum_{n=0}^{\infty} A_n\}$ (9)

Introduce the recursive relation in new modified Adomian Decomposition methods. Applying the inverse Laplace transform (9) in first part gives $y_0(x)$ and it is defined A_0.If using A_0 and will be evaluated of$y_1(x)$, then determination A_1 and that will allow determining of $y_2(x)$ and so on. In this expression by using equation (2) then we get $y_0(x) = L\{f(x)\}$

$y_{n+1}(x) = L\{k(x)\}L\{A_n\}, n \geq 0$ (10)

We applying in Laplace inverse transform methods and find *f(x)* can be denoted in another function Ψ as follows

$\Psi_0 = f(x)$ and by using in equation (10) in inverse Laplace transform methods and adding so find in approximate solution $\Psi = f_0(x) + f_1(x) + f_2(x) + \ldots\ldots\ldots + f_n(x)$ (11)

We define

$y = f_u(x) + \ldots\ldots\ldots + f_{u+v}(x)$ (12)

Where u= 0, 1, 2, 3, …… m , v= 0, 1, 2, ……..m-k.

After applying inverse transform and exact solution can be find as

$y_{n+1} = L^{-1}[L\{k(x)\}L\{A_n\}]\ ,\ n \geq 0$ (13)

Laplace Homotopy Perturpation method for Nonlinear Abel's Integral Equation:

We suppose a Abel's integral equation of second kinds

$y(x) = f(x) + \int_0^x \frac{F(y(t))}{\sqrt{x-t}} dt$ (14)

Applying the Laplace transform on both side in equation (14) then we get

$L[y(x)] = L[f(x)] + L\left[\int_0^x \frac{F(y(t))}{\sqrt{x-t}} dt\right]$ (15)

By using the Laplace transform convolution property that becomes

$L[y(x)] = L[f(x)] + \sqrt{\frac{\pi}{s}} L[y(x)]$ (16)

In above equation applying laplace inverse transform on both side, then we get

$y(x) = f(x) + L^{-1}\{\sqrt{\frac{\pi}{s}} L[y(x)]$ In Abel integral (14) has a solution of as series form

$\Psi(x) = \sum_{n=0}^{\infty} r^n\, \Psi_n(x)$ (17)

We use in the following iterative scheme, where $\Psi_n(x), n = 0, 1, 2, 3, \ldots\ldots\ldots$ are function to be determined.

By using HPTM in equation (14) and we consider in convex homotopy

$\sum_{n=0}^{\infty} r^n\, \Psi_n(x) = f(x) + r\left\{L^{-1}(\sqrt{\frac{\pi}{s}} L\left(\sum_{n=0}^{\infty} r^n\, \Psi_n(x)\right)\right\}$ (18)

Equating the coefficient of r on both sides, this is coupling of the Laplace and Homotopy method. Following approximation is obtained as:

$r^0: \Psi_0(x) = f(x)$

$$r^n: \Psi_n(x) = L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\Psi_{n-1}(x))\right\} \qquad n = 1,2,3, \dots \dots .... \tag{19}$$

Then solution of equation (14) is given as

$$y(x) = \lim_{r\to 1} \Psi(x) = \sum_{n=0}^{\infty} \Psi_n(x) \tag{20}$$

NUMERICAL RESULTS:

Example1. Singular Volterra Integral Equation

$$y(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt, \qquad 0 \le x \le 1. \tag{21}$$

Exact Solution $y(x) = x^2$.

Laplace MADM for Nonlinear Abel's Integral Equation:

$\Psi_0(x) = x^2$, $\Psi_1(x) = \frac{16}{15}x^{5/2}$, $\Psi_2(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\Psi_1(x))\right\}$

$\Psi_3(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(-\frac{\pi}{3}x^3)\right\} = \frac{32\pi}{105}x^{7/2}$, $\Psi_4(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\frac{32\pi}{105}x^{\frac{7}{2}})\right\} = -\frac{\pi^2}{24}x^4$

$\Psi_5(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(-\frac{\pi^2}{24}x^4)\right\} = \frac{32\pi^2}{945}x^{9/2}$. Hence the solution of the given problem of (21) is given as

$\Psi(x) = \Psi_0 + \Psi_1 + \Psi_2 + \dots \dots ....$

$\Psi(x) = x^2 + \frac{16}{15}x^{5/2} - \frac{\pi}{3}x^3 + \frac{32\pi}{105}x^{7/2} - \frac{\pi^2}{24}x^4 + \frac{32\pi^2}{945}x^{9/2}$

Laplace Homotopy Perturpation method for Nonlinear Abel's Integral Equation:

Homotopy Perturbation method can be constructed in following manners

$$\sum_{n=0}^{\infty} r^n\Psi_n(x) = x^2 + \frac{16}{15}x^{5/2} - r\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}}L(\sum_{n=0}^{\infty} r^n \Psi_n)\right)\right\} \tag{22}$$

Using in recurrence Form in equation (22) as follows

$\Psi_0(x) = x^2 + \frac{16}{15}x^{5/2}$

$\Psi_1(x) = -\frac{16}{15}x^{5/2} - \frac{\pi}{3}x^3$

$\Psi_2(x) = \frac{\pi}{3}x^3 + \frac{32\pi}{105}x^{7/2}$

$\Psi_3(x) = -\frac{\pi^2}{12}x^4 - \frac{32\pi}{105}x^{7/2}$ .Hence the solution of the given problem of (22) is given as

$$\Psi(x) = \Psi_0 + \Psi_1 + \Psi_2 + \dots \dots .... = x^2.$$

Numerical results we obtain Laplace MADM for Nonlinear Abel's Integral Equation on the following **Table 1**:

Node	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00	0.000000000000	0.000000000000
0.1	0.01	0.012599778805	0.002599778805
0.2	0.04	0.053707802750	0.013707802750
0.3	0.09	0.126610682700	0.036610682700
0.4	0.16	0.234540844000	0.074540844000
0.5	0.25	0.381329017200	0.131329017200
0.6	0.36	0.571654256400	0.211654256400
0.7	0.49	0.811205632000	0.321205632000
0.8	0.64	1.106808221000	0.466808221000
0.9	0.81	1.466531012000	0.656531012000
1.0	1.00	1.899784040000	0.899784040000

Example2. Singular Volterra Integral Equation

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt, \qquad 0 \le x \le 1 \qquad (24)$$

Exact Solution $y(x) = x$.

Laplace MADM for Nonlinear Abel's Integral Equation:

$$\Psi_0(x) = x, \Psi_1(x) = \frac{4}{3}x^{3/2}, \Psi_2(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\Psi_1(x))\right\}$$

$$\Psi_2(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\frac{4}{3}x^{\frac{3}{2}})\right\} = -\frac{\pi}{2}x^2, \Psi_3(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(-\frac{\pi}{2}x^2)\right\} = \frac{8\pi}{15}x^{5/2}$$

$$\Psi_4(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\frac{8\pi}{15}x^{\frac{5}{2}})\right\} = -\frac{\pi^2}{6}x^3, \Psi_5(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(-\frac{\pi^2}{6}x^3)\right\} = \frac{16\pi^2}{105}x^{7/2}$$

$\Psi_6(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\frac{16\pi^2}{105}x^{7/2})\right\} = -\frac{\pi^3}{24}x^4$ .Hence the solution of the given problem of (24) is given as

$$\Psi(x) = \Psi_0 + \Psi_1 + \Psi_2 + \ldots\ldots\ldots$$

$$\Psi(x) = x + \frac{4}{3}x^{3/2} - \frac{\pi x^2}{2} + \frac{8\pi}{15}x^{5/2} - \frac{\pi^2}{6}x^3 + \frac{16\pi^2}{105}x^{7/2} - \frac{\pi^3}{24}x^4.$$

Laplace Homotopy Perturpation method for Nonlinear Abel's Integral Equation:

Homotopy Perturbation method can be constracted in following manners

$$\sum_{n=0}^{\infty} r^n \Psi_n(x) = x + \frac{4}{3}x^{3/2} - r\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}}L(\sum_{n=0}^{\infty} r^n \Psi_n)\right)\right\} \qquad (25)$$

Using in recurrence Form in equation (25) as follows

$$\Psi_0(x) = x + \frac{4}{3}x^{3/2}, \Psi_1(x) = -\frac{4}{3}x^{3/2} - \frac{\pi}{2}x^2, \Psi_2(x) = \frac{\pi}{2}x^2 + \frac{8\pi}{15}x^{5/2}$$

$\Psi_3(x) = -\frac{8\pi}{15}x^{5/2} - \frac{\pi^2}{6}x^3$ Hence the solution of the given problem of (25) is given as

$\Psi(x) = \Psi_0 + \Psi_1 + \Psi_2 + \ldots\ldots\ldots. = x$

Numerical results we obtain Laplace MADM for Nonlinear Abel's Integral Equation on the following **Table 2**:

Node	Exact Solution	Approximate Solution	Absolute Error
0.0	0.0	0.0000000000	0.000000000000
0.1	0.1	0.1304503592	0.0304503592
0.2	0.2	0.2765297632	0.0765297632
0.3	0.3	0.4276210063	0.1276210063
0.4	0.4	0.5779499519	0.1779499519
0.5	0.5	0.7212712037	0.2212712037
0.6	0.6	0.8499799110	0.2499799110
0.7	0.7	0.9547658097	0.2547658097
0.8	0.8	1.0605801710	0.2605801710
0.9	0.9	1.0659378300	0.1659378300
1.0	1.0	1.0034985430	0.0034985430

Conclusion: In this paper, we applied Laplace transform in Modified Adomian Decomposition method &Homotopy Perturbation method to solve singular integral equation. The proposed method has been applied to solve several number of singular volterra integral equations of second kinds. The obtained solutions, in comparison with exact solutions and approximate solutions admit a remarkable accuracy. This results is very fast and lower approximations can be achieve high accuracy.

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