

A Transformation Method to Construct Exactly Solvable power-law potentials

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Abstract

Exact analytic Quantum Mechanical bound state and scattering state solutions of Schrodinger Green's Function (GF) equation for Double anharmonic oscillator in five dimensions are given. By applying extended transformation method we have constructed exact solution of D -dimensional radial Schrodinger GF equation for a set of power-law multi-term potentials taking $V(r) = ar^2 + br + cr^{-1}$ as input reference. The restriction on the parameters of the given potentials and angular momentum quantum number l are obtain. We discuss the normalizability of the bound state solution of a generated exactly solvable potential.

Keywords: Extended transformation, Exactly solvable potential, Schrodinger Green's function.

1.Introduction

In the recent years there is fresh interest in finding exact solution of Schrodinger Green's function (GF) equation. The exact solution of the Schrodinger GF equation with some well-known central and non-central potential is an important subject in quantum mechanical (QM) problem. These facilitates both parturbative and non-parturbative methods, in bound state and scattering state regime. For the success of the application of various approximation schemes it is necessary that an exactly solvable potential may be differs from the given potential as little as possible.

Considerable effort has been made towards obtaining exact solution of the Schrodinger equation for potential of physical interest [1-6]. Effort has been made to solve the time-independent Schrodinger equation for central multi-term potential containing negative powers of radial coordinate in two as well as three dimensions [7-16]. However, analysis of such problems in arbitrary D -dimensions is limited to the best of our knowledge [15-18] The success in one and three dimensional cases can be termed partial as only a few states can be found analytically [1]. In this paper we have use a simple and compact mapping procedure called the extended transformation (ET) method [19-23] to obtain exact analytic solutions of the anharmonic oscillator in 5-dimensional space, taking as starting point of exactly solved 3-dimensional QM problem with central fractional power singular potential [24] and Construct ESP from central Harmonic +Linear+ Coulomb potential within the framework of GF technique, in any pre-assigned dimensional space. Our main objective is to construct more exactly solvable potentials (ESPs) from the reference potentials and to show their hierarchical connections. In this transformation method the normalizability of the eigenfunctions of the generated QS can easily

be verified in most cases. We discuss a procedure to regroup this set of energy-dependent Sturmian to a normal/physical QS.

The paper is organized as follows. In Section 2, we briefly review the ET method. In Section 3, we apply ET on power law potentials for generating new QSs. Finally, the important results and conclusions are discussed in section 4.

2. FORMALISM

Spherically symmetric central potential $V(r)$ satisfies the differential equation ($\hbar = 1 = 2m$)

$$\left[\frac{\partial}{\partial t} + \nabla^2 - V(r) \right] \Omega(\vec{r}t, \vec{r}_0t_0) = \frac{i\delta(\vec{r}, \vec{r}_0)\delta(t, t_0)}{r_0^{D-1}} \quad (1)$$

The propagator $\Omega(\vec{r}t, \vec{r}_0t_0)$ of a particle moving from an initial position \vec{r}_0 at time t_0 to a final position \vec{r} at time t . It is convenient for our purpose to deal with energy and angular momentum dependent GF $G(\vec{r}, \vec{r}_0, E, V(r))$. For this we Fourier transform as

$$G(\vec{r}, \vec{r}_0, E) = i \int_0^\infty e^{iEnT} \Omega(\vec{r}t, \vec{r}_0t_0, V(r)) dT \quad (2)$$

where $T = t - t_0$. Which satisfies the Schrodinger GF equation

$$[E + \nabla^2 - V(r)]G(\vec{r}, \vec{r}_0, E) = \frac{\delta(r - r_0)}{r_0^{D-1}} \quad (3)$$

It is followed by a partial wave expansion

$$G(\vec{r}, \vec{r}_0, E) = \sum_{l=0}^\infty \sum_{m=-l}^l G_l(\vec{r}, \vec{r}_0) Y_l^m(\theta, \phi) Y_l^{*m}(\theta_0, \phi_0) \quad (4)$$

We now determine the radial GF $G_l(\vec{r}, \vec{r}_0, E)$, as we shall be dealing with Central potential. The radial GF is found to satisfies the D_A -dimensional radial Schrodinger GF equation

$$\left[\partial_r^2 + \frac{D_A-1}{r} \partial_r + E_A - V_A(r) - \frac{l_A(l_A-D_A-2)}{r^2} \right] G_{l_A}(r, r_0, E_A) = \frac{\delta(r-r_0)}{r_0^{D_A-1}} \quad (5)$$

where r and r_0 are the dimensionless spatial coordinates for The corresponding integral equation is $\varphi_A(r) = \int G_A(r, r_0; E_A)(E_A - V_A(r))\varphi_A(r_0)r_0^{D_A-1} d r_0$, where the GF and energy eigenvalues E_A are known for the given potential $V_A(r)$.

The completeness of the set of energy eigenfunctions allows us to have eigenfunction expansion of the energy GF as:

$$G_A(r, r_0; E_A) = \sum_{n=0}^{\infty} \frac{\varphi_A(r)\varphi_A^*(r_0)}{E - E_A - i\epsilon} \tag{6}$$

from which we read off the analytic form of the wave function of the solved quantum system.

We perform the ET [22] to equation (5) which comprises of coordinate transformation $r \rightarrow g_B(r)$, $r_0 \rightarrow g_B(r_0)$ and a functional transformation

$$G_{l_B}(r, r_0; E_B) = f_B^{-1}(r)G_{l_A}(g_B(r), g_B(r_0))f_B^{-1}(r_0) \tag{7}$$

that the resulting equation is of the same form as (5) but with new parameters and leads to the following equation

$$\left[\partial_r^2 + \left(\frac{d}{dr} \ln \frac{f_B^2(r)g_B^{D_A-1}(r)}{g'_B(r)} \right) \partial_r + \left(\frac{d}{dr} \ln f_B(r) \right) \left(\frac{d}{dr} \ln \frac{f'_B g_B^{D_A-1}(r)}{g'_B(r)} \right) + g_B'^2(E_A - V_A(g_B)) - \frac{l_A(l_A - D_A - 2)}{g_B^2} \right] G_{l_B}(r, r_0; E_A) = g_B'^2 f_B^{-1}(r) \frac{\delta(g_B(r) - g_B(r_0))}{g_B^{D_A-1}(r_0)} f_B^{-1}(r_0) \tag{8}$$

The transformation functions $g_B(r)$ and $g_B(r_0)$ are smooth differentiable function, which are at least three times differentiable function and $f_B^{-1}(r)$, $f_B^{-1}(r_0)$ are the modulating function required to mould the above equation to the standard Schrodinger GF equation form. We make the coefficient of the first order derivative equal to $\frac{D_B-1}{r}$, fixing the functional form of

$$f_B^{-1}(r) = C_N g_B'^{\frac{1}{2}}(r) g_B^{\frac{D_A-1}{2}}(r) r^{-\frac{D_B-1}{2}}(r) \tag{9}$$

where C_N is the normalization constant. Therefore equation (7) and (9) lead to

$$G_{l_B}(r, r_0; E_B) = g_B'^{\frac{1}{2}}(r) g_B^{\frac{D_A-1}{2}}(r) r^{-\frac{D_B-1}{2}} G_{l_A}(g_B(r), g_B(r_0), g_B'^2 E_A) g_B'^{\frac{1}{2}}(r_0) g_B^{\frac{D_A-1}{2}}(r_0) r_0^{-\frac{D_B-1}{2}}(r_0) \tag{10}$$

The right hand side of equation (8) can be simplified to $\frac{\delta(r-r_0)}{r_0^{D_B-1}}$. Which changes equation (8) to

$$\left[\partial_r^2 + \frac{D_B - 1}{r} \partial_r + \frac{1}{2} \frac{g_B'''}{g_B'} - \frac{3}{4} \left(\frac{g_B''}{g_B'} \right)^2 - \frac{D_A - 1}{2} \frac{D_A - 3}{2} \left(\frac{g_B'}{g_B} \right)^2 + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2} + g_B'^2 \times \left(E_A - V_A(g_B(r)) - \frac{\left(l_A - \frac{D_A}{2} - 1 \right)^2}{g_B^2} + \frac{(D_A - 2)^2}{4g_B^2} \right) \right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B-1}} \tag{11}$$

In case of multi-term A-QS, we have to select a term of $V_A(g_B(r))$ as working potential (WP) to implement ET and is designated as $V_A^W(g_B(r))$.

In order mould equation (11) to the standard form of the Schrodinger GF equation, the following plausible ansatze have to be made, which are integral part of the transformation method.

$$g_B'^2 V_A^W(g_B(r)) = -E_B \tag{12}$$

$$V_B(r) = -g_B'^2 (E_A - V_A(g_B(r)) + V_A^W(g_B(r))) \tag{13}$$

$$\frac{g_B'^2 \left(l_A + \frac{D_A}{2} - 1 \right)}{g_B^2} = \frac{\left(l_B + \frac{D_B}{2} - 1 \right)^2}{r^2} \tag{14}$$

Invoking the ansatze (12) to (14), equation (11) becomes

$$\left[\partial_r^2 + \frac{D_B - 1}{r} \partial_r + \frac{1}{2} \frac{g_B'''}{g_B'} - \frac{3}{4} \left(\frac{g_B''}{g_B'} \right)^2 - \frac{D_A - 1}{2} \frac{D_A - 3}{2} \left(\frac{g_B'}{g_B} \right)^2 + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2} + \left(E_B - V_B(r) - \frac{\left(l_B - \frac{D_B}{2} - 1 \right)^2}{r^2} - \frac{(D_A - 2)^2}{4g_B^2} \left(\frac{g_B'}{g_B} \right)^2 \right) \right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}} \tag{15}$$

In the case of a power law type potential $V_A(r) = a_A r^{b_A}$, the dimension of the Euclidean space to which the transformed system gets transported, can be arbitrarily pre-assigned only when ET is performed and the expression $\frac{1}{2} \frac{g_B'''}{g_B'} - \frac{3}{4} \left(\frac{g_B''}{g_B'} \right)^2 - \frac{D_A - 1}{2} \frac{D_A - 3}{2} \left(\frac{g_B'}{g_B} \right)^2 + \frac{D_B - 1}{2} \frac{D_B - 3}{2} \frac{1}{r^2} - \frac{\left(l_B - \frac{D_B}{2} - 1 \right)^2}{r^2} + \frac{(D_A - 2)^2}{4g_B^2} \left(\frac{g_B'}{g_B} \right)^2$ in equation (15) is reduce to the correct form of the centrifugal barrier term, $\frac{l_B(l_B - D_B - 2)}{r^2}$ [23], in D_B -dimensional space. The transformed B-QS Schrodinger GF equation is established as

$$\left[\partial_r^2 + \frac{D_B - 1}{r} \partial_r + E_B - V_B(r) - \frac{l_B(l_B - D_B - 2)}{r^2} \right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}} \tag{16}$$

In equation (12) $V_A(g_B(r))$ is termed as WP. WP can be chosen in $2^n - 1$ different ways. In fact we can pick any number of terms of the multi-term potential, the least being a single term and designate it as the WP. Let for simplicity i^{th} term as WP, in which $V_A^W(g_B(r)) = a_i g^{b_i}$. Ansatz (12) is now $g_B'^2 a_i g^{b_i} = -E_B$, and with a simple integration

$$g_B(r) = \left[\pm \frac{b_i + 2}{2} \sqrt{-\frac{E_B}{a_i} r + C} \right]^{\frac{2}{b_i+2}} \tag{17}$$

where C is the integration constant. For power law $g_B(r)$, g'_B is also a power law function of r ; hence by equation (13), one gets a power law $V_B(r)$. The energy eigenvalues E_B of the B-QS is simply obtained by putting the coefficient of the r -independent part of $V_B(r)$ -which would be a product of a function $F(E_B)$ and E_A -equal to the characteristic constant of the B system. E_B is specified in terms of the known E_A of the parent system (A-system). However the quantum number n and N are different as l_A and l_B are in general different. The relation between the angular momentum quantum numbers are obtained from equation (14) and is

$$4l_A = (b_A + 2)(2l_B + D_B - 2)(D_A - 2) \tag{18}$$

From equation (6) and (10), the eigenfunction expansion of B-QS Green's function is

$$\begin{aligned} G_{l_B}(r, r_0; E_B) &= \sum_{n=0}^{\infty} \frac{\left(\frac{g_B^{D_A-1}(r)}{g'_B(r)r^{D_B-1}} \right)^{\frac{1}{2}} \varphi_{l_A}(g_B(r)) \varphi_{l_A}^*(g_B(r_0)) \left(\frac{g_B^{D_A-1}(r_0)}{g'_B(r_0)r_0^{D_B-1}} \right)^{1/2}}{E - \epsilon_n^B - i\epsilon} \\ &= \sum_{N=0}^{\infty} \frac{\varphi_{l_B}^{(N)}(r) \varphi_{l_B}^{*(N)}(r_0)}{E - \epsilon_n^B - i\epsilon} \end{aligned} \tag{19}$$

The B-QS eigenfunctions $\varphi_B^{(N)}(r)$ can be read off from equation (19) in conjunction with equation (9) as

$$\varphi_{l_B}(r) = C_N \left(\frac{g_B^{D_A-1}(r)}{g'_B(r)r^{D_B-1}} \right)^{\frac{1}{2}} \varphi_{l_A}(g_B(r)) \tag{20}$$

3. APPLICATION OF ET METHOD ON POWER-LAW POTENTIAL

3.1. Construction of EAS from Fractional power singular potential

We consider the 3-dimensional quantum mechanical problem with central Fractional power singular potential [24] as a typical representative of QS with power law potential. Let it be denoted as our A-QS. The potential is

$$V_A(r) = ar^{-4/3} + br^{2/3} + cr^{-2/3} \tag{21}$$

Where a , b and c are the parameters of the potential. The constraint equation for the parameters is

$$c = \left(2l_A + \frac{5}{3}\right) \left[\left(2l_A + \frac{7}{3}\right) \sqrt{a} + b\right] \quad (22)$$

which satisfies the radial Schrodinger GF equation in three-dimensional space

$$\left[\partial_r^2 + \frac{2}{r} \partial_r + E_A - (ar^{-4/3} + br^{2/3} + cr^{-2/3}) - \frac{l_A(l_A+1)}{r^2}\right] G_{l_A}(r, r_0, E_A) = \frac{\delta(r-r_0)}{r_0^2} \quad (23)$$

The exact eigenfunctions and eigenvalues of the potential are given as

$$\varphi_A(r) = N_A r^{l_A} \exp\left(-\frac{3}{4} \sqrt{a} r^{4/3} + \frac{3}{4} \frac{E_A}{\sqrt{a}} r^{2/3}\right) \quad (24)$$

$$E_A = \pm \sqrt{4a} \left[\left(2l_A + \frac{7}{3}\right) \sqrt{a} + b\right]^{1/2} \quad (25)$$

Where N_A is the normalization constant for A-QS.

Employing ET to equation (23), one obtains the Schrodinger GF equation for the new QS (B-QS) in D_B -dimensional space:

$$\left[\partial_r^2 + \frac{D_B-1}{r} \partial_r + \frac{1}{2} \frac{g_B'''}{g_B} - \frac{3}{4} \left(\frac{g_B''}{g_B}\right)^2 + \frac{D_B-1}{2} \frac{D_B-3}{2} \frac{1}{r^2} + g_B'^2 \left(E_A - (a g_B^{-4/3} + b g_B^{2/3} + c g_B^{-2/3}) - \frac{l_A(l_A+1)}{g_B^2}\right)\right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}} \quad (26)$$

Here $V_A(r)$ is a three term potential, as given in equation (21). From this multi-term potential of A-QS, the WP can be chosen in $(2^3 - 1)$ different ways.

Taking $a g_B^{-4/3}$ as working potential, the ansatz (12) required to make a standard Schrodinger GF equation of B-QS now becomes

$$g_B'^2 a g_B^{-4/3} = -E_B \quad (27)$$

Integrating equation (27) yields

$$g_B(r) = \pm \left(-\frac{E_B}{9a}\right) r^3 + C \quad (28)$$

where $C = 0$ makes the transformation factor $g_B(r)$ to the desired local property $g_B(0) = 0$, this makes $g_B(r)$ power law. Taking the positive sign in equation (28) and utilizing equation (13), we get following B- system quantum potential:

$$V_B(r) = \alpha^2 r^6 + \beta r^4 + \gamma r^2 \tag{29}$$

The coefficients α, β and γ required for complete specification of $V_B(r)$ are given by $\alpha^2 = \frac{a}{9} \left(\frac{E_B}{3a}\right)^4$, $\beta = b \left(\frac{E_B}{3c}\right)^2$ and $\gamma = \frac{E_A}{3} \left(\frac{E_B}{3c}\right)^3$.

and after Langer modification [25]

$$\frac{g_B'^2 \left(l_A + \frac{1}{2}\right)}{g_B^2} = \frac{\left(l_B + \frac{D_B}{2} - 1\right)^2}{r^2} \tag{30}$$

where

$$D_B = 1 + r \frac{d}{dr} \ln \frac{g_B^2}{g_B'} \tag{31}$$

is the dimension of space into which the transformed system will be transported to automatically. And the relationship between the angular momentum quantum number l_A and l_B coming from equation (30) as

$$l_B = 3l_A \tag{32}$$

Under the co-ordinate transformation the roles of the constraint (22) and energy eigenvalues (25) are interchanged to give the energy eigenvalues of B-QS

$$E_B = (2l_B + 5)[(2l_B + 7)\alpha + \beta]^{1/2} \tag{33}$$

and the constraint equation is

$$\gamma = \pm 2\alpha[(2l_B + 7)\alpha + \beta]^{1/2} \tag{34}$$

Equation (23) then becomes

$$\left[\partial_r^2 + \frac{4}{r} \partial_r + \left\{ (2l_B + 5)[(2l_B + 7)\alpha + \beta]^{1/2} - (\alpha^2 r^6 + \beta r^4 + \gamma r^2) - \frac{l_B(l_B + 5 - 2)}{r^2} \right\} \right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B - 1}} \tag{35}$$

which is radial Schrodinger GF equation in 5-dimensional space.

Eigenfunction expansion of B-QS Green's function is given by equation (19) which is

$$G_{l_B}(r, r_0; E_B) = \sum_{N=0}^{\infty} \frac{r^{l_B} \exp\left(-\frac{1}{4}\alpha r^4 - \frac{1}{4}\frac{\beta}{\alpha}r^2\right) r_0^{l_B} \exp\left(-\frac{1}{4}\alpha r_0^4 - \frac{1}{4}\frac{\beta}{\alpha}r_0^2\right) \varphi_B^{*(N)}(r_0)}{E - E_B - i\epsilon} \quad (36)$$

from which B-QS energy eigenfunctions $\varphi_{l_B}(r)$ can be read off. In view of equation (24) the explicit form of the new exact energy eigenfunction becomes:

$$\varphi_{l_B}(r) = N_B r^{l_B} \exp\left(-\frac{1}{4}\alpha r^4 - \frac{1}{4}\frac{\beta}{\alpha}r^2\right) \quad (37)$$

The Normalization constant N_B of the wavefunction $\varphi_{l_B}(r)$ is

$$N_B \propto \left[\int_0^{\infty} \varphi_{l_A}(g_B(r)) g_B'^{-2} \varphi_{l_A}^*(g_B(r_0)) \right]^{-\frac{1}{2}} \quad (38)$$

In view the fact that $g_B(0) = 0$ and $g_B(\infty) = \infty$. The measure of integration the Euclidean space of $\varphi_{l_B}(r)$ is always proportional to $\frac{g_B^2}{g_B'^2}$ and yield, $N_B \propto \langle r^{-4/3} \rangle$.

By equation (27) $\langle r^{-4/3} \rangle$ always exists as it is the expectation value of a part of the potential belonging to genuine quantum mechanical system i.e., the original system.

It is interesting to note that the constraint equation as well as the energy eigenvalues of 1-dimensional problem of Fleases [1] are same as our relation for the 5-dimensional case with $l_B = 0$.

In scattering states the energy of the scattered particle is positive and continuous, hence $E_B = k^2$, we consider r to be so large that $V_B(r)$ and l_B term in equation (35) can be neglected and write the analogue of equation (36) as:

$$G_B(r, r_0; k^2) = \int_0^{\infty} \frac{dE}{E - k^2 - i\epsilon} r \exp\left(-\frac{1}{4}\alpha r^4 - \frac{1}{4}\frac{\beta}{\alpha}r^2\right) r_0 \exp\left(-\frac{1}{4}\alpha r_0^4 - \frac{1}{4}\frac{\beta}{\alpha}r_0^2\right) \quad (39)$$

Using symbolic identity $\lim_{r \rightarrow \infty} \frac{1}{(E - k^2) \pm i\epsilon} P \frac{1}{E - k^2} \mp i\pi\delta(E - k^2)$ we can define imaginary part of $G_B(r, r_0; k^2)$ and the scattering wave function

$$\varphi_B^{scatt}(r) = r \exp\left(-\frac{1}{4}\alpha r^4 - \frac{1}{4}\frac{\beta}{\alpha}r^2\right) \quad (40)$$

3.2. Construction of EAS from Harmonic +Linear+ Coulomb potential

In this section we applied our formalism on an exactly solved Central Coulomb +Linear+ Harmonic potential. The potential has the form

$$V_A(r) = ar^2 + br + cr^{-1} \tag{37}$$

where a , b and c are the parameters of the potential. The constraint equation for the parameters and l_A is given by

$$b + (2l_A + 2) \frac{c}{2\sqrt{a}} = 0 \tag{38}$$

The exact eigenfunctions and eigenvalues of the potential are given [24] as

$$\varphi_{l_A}(r) = N_A r^{l_A} \exp\left(-\frac{1}{2}\sqrt{a}r^2 - \frac{1}{2}\frac{c}{\sqrt{a}}r\right) \tag{39}$$

Where N_A is the normalization constant for A-QS.

$$E_A = -\left[\frac{c^2}{4a} - (2l_A + 3)\sqrt{a}\right] \tag{40}$$

Corresponding to $D_A = 1$, dimensional differential equation is

$$\left[\partial_r^2 + E_A - (ar^2 + br + cr^{-1}) - \frac{l_A(l_A + 1)}{r^2}\right]G_{l_A}(r, r_0; E_A) = \delta(r - r_0) \tag{41}$$

Here $V_A(r)$ is a three term potential. From this multi-term potential of A-QS, the WP can be chosen in $(2^3 - 1)$ different ways. To implement ET on A-QS, as a specific choice, we select ar^2 as the WP. The functional form of $g_B(r)$, obtained from (14), by an integration as

$$g_B(r) = \pm \frac{(-E_B)^{1/2}}{a} \sqrt{2r} \tag{42}$$

with the local property $g_B(0) = 0$. Taking the positive sign in equation (42) and utilizing equation (13), we get the following B-Sturmian quantum system potential:

$$V_{B_1}(r) = \alpha_1 r^{-1} + \alpha_2^{(n)} r^{-1/2} + \alpha_3^{(n)} r^{-3/2} \tag{43}$$

with $\alpha_1 = \frac{1}{2}\left(-\frac{E_B}{a}\right)^{1/2}(-E_A)$, $\alpha_2^{(n)} = \frac{1}{\sqrt{2}}\left(-\frac{E_B}{a}\right)^{3/4}c$ and $\alpha_3^{(n)} = \frac{1}{2\sqrt{2}}\left(\left(-\frac{E_B}{a}\right)^{1/4}\right)b$

To make the above potential normal, we take $a \rightarrow a_n$, $b \rightarrow b_n$ and $c \rightarrow c_n$. Consequently, the normal form of $V_{B_1}(r)$ is given by

$$V_{B_1}(r) = \alpha_1 r^{-1} + \alpha_2 r^{-\frac{1}{2}} + \alpha_3 r^{-\frac{3}{2}} \tag{44}$$

The energy eigenvalues of B-QS is

$$E_B = -\left[\frac{\alpha_2}{4\alpha_3}(3 - 4l_B - 2D_B)\right]^2 \tag{45}$$

The angular momentum quantum number l_B of B-QS is related to the angular momentum quantum number l_A of A-QS through equation (18) and is

$$4\alpha_3^2 + \frac{\alpha_2}{4\alpha_3} (4l_B + 2D_B - 3)^3 (2l_B + D_B - 1) - \alpha_1 (4l_B + 2D_B - 3)^2 = 0 \tag{46}$$

Invoking the equations (44) and (45) in equation (16) we get the standard Schrodinger GF equation in D_B -dimensional Euclidean space as

$$\left[\partial_r^2 + \frac{D_B - 1}{r} \partial_r + E_B - \left(\alpha_1 r^{-1} + \alpha_2 r^{-\frac{1}{2}} + \alpha_3 r^{-\frac{3}{2}} \right) - \frac{l_B(l_B - D_B - 2)}{r^2} \right] G_{l_B}(r, r_0; E_B) = \frac{\delta(r - r_0)}{r_0^{D_B-1}} \tag{47}$$

From equation (19), the new radial eigenfunctions can be read off as

$$\varphi_{l_B}(r) = C_N r^{l_B} \left[-(-E_B)^{1/2} r - \frac{\alpha_2}{(-E_B)^{1/2}} r^{1/2} \right] \tag{48}$$

In a similar way, choosing the terms br and cr^{-1} as the WP (equation (37)) and then applying ET (equations (12) to (14)), we can generate two more exactly solved new QSs. The new generated potentials, energy eigenvalues and parameter constraints are summarized in table 1 and transformation function as well as energy eigenfunctions are given in table 2.

Table 1. List of ESP, E_B and constraint equation from Harmonic +Linear+ Coulomb potential $V_A(r) = ar^2 + br + cr^{-1}$ taking the WPs br and cr^{-1} .

$V_A^{(W)}(r)$	$V_B(r)$	E_B	Constraint equation
br	$V_{B_2}(r) = \beta_1 r^{-\frac{2}{3}} + \beta_2 r^{-\frac{4}{3}} + \beta_3 r^{\frac{2}{3}}$ $\beta_1 = \left(\frac{3}{2}\right)^{-\frac{2}{3}} \left(-\frac{E_B}{b}\right)^{\frac{2}{3}} (-E_A)$ $\beta_2 = \left(\frac{3}{2}\right)^{-\frac{4}{3}} \left(-\frac{E_B}{b}\right)^{\frac{1}{3}} c$ $\beta_3 = \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(-\frac{E_B}{b}\right)^{\frac{4}{3}} a$	$2\sqrt{\beta_3} \left[\left(2l_B + D_B - \frac{2}{3}\right) \sqrt{\beta_3} + \beta_1 \right]^{\frac{1}{2}}$	$\left(2l_B + D_B - \frac{4}{3}\right) \left[\left(2l_B + D_B - \frac{2}{3}\right) \sqrt{\beta_3} + \beta_1 \right]^{\frac{1}{2}} - \beta_2 = 0$
cr^{-1}	$V_{B_3}(r) = \gamma_1 r^2 + \gamma_2 r^4 + \gamma_3 r^6$ $\gamma_1 = \frac{1}{4} \left(-\frac{E_B}{c}\right)^2 (-E_A)$ $\gamma_2 = \frac{1}{16} \left(-\frac{E_B}{c}\right)^3 b$ $\gamma_3 = \frac{1}{64} \left(-\frac{E_B}{c}\right)^4 a$	$\frac{\gamma_2}{2\gamma_3} (2l_B + D_B)$	$\frac{\gamma_2^2}{4\gamma_3} - \sqrt{\gamma_3} (2l_B + D_B + 2) - \gamma_1 = 0$

Table 2. List of transformation functions $g_B(r)$, wave functions $\varphi_B(r)$ and scattering wave generated from Harmonic +Linear+ Coulomb potential, taking the WP as given in the Table 1.

Sl No	$g_B(r)$	$\varphi_{l_B}(r)$	$\varphi_B^{sctt}(r)$
1	$\left(\frac{3}{2}\right)^{\frac{2}{3}} \left(-\frac{E_B}{b}\right)^{\frac{1}{2}} r^{2/3}$	$r^{l_B} \exp \left[-\frac{3}{4} \sqrt{\beta_3} r^{\frac{4}{3}} + \frac{3}{4} \frac{E_B}{\sqrt{\beta_3}} r^{\frac{2}{3}} \right]^{1/2}$	$r \exp \left[-\frac{3}{4} \sqrt{\beta_3} r^{\frac{4}{3}} + \frac{3}{4} \frac{k^2}{\sqrt{\beta_3}} r^{\frac{2}{3}} \right]^{1/2}$
2	$\frac{1}{4} \left(-\frac{E_B}{c}\right) r^2$	$r^{l_B} \exp \left[-(-E_B)^{1/2} r - \frac{\gamma_3}{(-E_B)^{1/2}} r^{\frac{1}{2}} \right]$	$r \exp \left[-ikr - \frac{\gamma_3}{ik} r^{\frac{1}{2}} \right]$

4. DISCUSSION AND CONCLUSION

We have constructed a class of exactly solved quantum system in non-relativistic quantum mechanics using ET method in any arbitrary dimensional Euclidean spaces within the framework of GF techniques. In this succinct report in sec. 3.1 we present the ET method to obtain exact analytic bound state and scattering state solutions of anharmonic oscillator in 5-dimensional Euclidean space, taking as 3-dimensional quantum mechanical problem with central fractional power potential, as a typical representative power law potential. It is interesting to note that, the structure of the constraint equation as well as the energy eigenvalues of 1-dimensional problem of Flesses [1] is same as our relations for the 5-dimensional case with $l_B = 0$. The solution of the Schrodinger GF equation with the anharmonic oscillator potential can be applied in the field of fiber optics, where one wants to solve a similar problem of an inhomogeneous spherical or circular wave guide with refractive index profile function of the doubly an harmonic sextic-type potential. Its solution is also applicable to molecular physics. Again this solutions help in understanding some aspects of modern field theories including false vacua. In section 3.2 we have generated 3 different exactly solved QSs by taking mixture potential as input reference potential, selecting WP differently.

For quantum multi-term potentials it is possible to generate a finite number of different exactly solved quantum systems by selecting WPs differently. However we restrict ourselves to taking one term WP. Multi-term WPs usually avoided as they offer practical difficulties: as the integrals specifying the transformation function $g_B(r)$ cannot be extracted analytically in most of the cases, even if such integrals are found they are of the form $F(g_B(r)) = r + C$ and the analytical inverse function F^{-1} cannot be found. The ET method may be applied repeatedly by selecting WP differently to generate a variety of solved QSs. Successive application of this method to generated QS will revert it back to the parent QS. It is noteworthy that under ET the constraint equation gets converted in to the energy eigenvalue expression and the energy eigenvalue in to constraint equation. A very useful property of the ET method the wave functions of the generated QSs are almost always normalizable. Explicit expressions for energy eigenvalues and eigenfunctions of various potentials are found.

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