A TWO SPECIES MODEL OF NEUTRALISM WITH MORTALITY RATE FOR THE FIRST SPECIES

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***Abstract*:** In this paper, the system comprises of two neutral species S_1 and S_2 with mortality rate for the first species. Here both the two species S_1 and S_2 have limited resources. The model equations constitute a set of two first order non-linear simultaneous differential equations. Criteria for the asymptotic stability of all the four equilibrium states are established and the trajectories of the perturbations over the equilibrium states are illustrated. Further, the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

***Key words*:** Characteristic equation, Equilibrium State, Neutralism, Trajectories, Stable, Unstable.

***2010 Mathematics Subject Classification*:** 92D25, 92D40

1. INTRODUCTION:

Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment sustain themselves on common resources. It is a common observations that the species of same nature can not flourish is isolation without any interaction with species of different kinds. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [20]. Syn-ecology is an ecosystem comprised of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation, Parasitism and so on.

Mathematical Modeling plays a vital role in providing insight into the mutual relationships (positive, negative) between the interacting species. Several authors Ma [6], Moghadas[7], Murray [8] and Sze-Bi Hsu [22] were introduced the general concepts of Modeling in Biological Science. Srinivas [21] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [9-19] investigated continuous and discrete models on two, three and four species syn-ecosystems.

2. BASIC EQUATIONS OF THE MODEL:

The model equations for the two species neutralism are given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation Adopted:

$N_i(t)$	: The population strength of S_i at time t, $i=1,2$
t	: Time instant.
d_1	: Natural death rate of S_1
g_2	: Natural growth rate of S_2
a_{ii}	: Self inhibition coefficients of S_i, $i=1,2$
$e_1 = \frac{d_1}{a_{11}}$	: Extinction coefficient of S_1
$k_2 = \frac{g_2}{a_{22}}$	: Carrying capacity of S_2

Further the variables N_1, N_2 are non-negative and the model parameters $d_1, g_2, a_{11}, a_{22}, e_1, k_2$ are assumed to be non-negative constants.

Equation for the first species (N_1):

$$\frac{dN_1}{dt} = -\left(d_1 + a_{11}N_1\right)N_1 \tag{1}$$

Equation for the second species (N_2):

$$\frac{dN_2}{dt} = \left(g_2 - a_{22}N_2\right)N_2 \tag{2}$$

3. EQUILIBRIUM STATES:

The system under investigation has four equilibrium states given by

$$\frac{dN_i}{dt} = 0, i = 1, 2 \tag{3}$$

Fully washed out state.

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0$$

States in which only one of the two species is washed out while the other one is not.

$$E_2 : \bar{N}_1 = -e_1, \bar{N}_2 = 0; \quad E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2$$

The co-existent state (or) normal steady state.

$$E_4 : \bar{N}_1 = -e_1, \bar{N}_2 = k_2$$

4. STABILITY OF THE EQUILIBRIUM STATES:

Let $N = \left(N_1, N_2\right) = \bar{N} + U$, (4)

where $U = \left(u_1, u_2\right)^T$ is a small perturbation over the equilibrium state $\bar{N} = \left(\bar{N}_1, \bar{N}_2\right)$.

The basic equations (1) and (2) are quasi-linearized to obtain the equations for the perturbed state as,

$$\frac{dU}{dt} = AU \ , \quad (5)$$

with $A(E) = \begin{bmatrix} -d_1 - 2a_{11}\bar{N}_1 & 0 \\ 0 & g_2 - 2a_{22}\bar{N}_2 \end{bmatrix}$ (6)

The characteristic equation for the system is $|A - \lambda I| = 0$ (7)

The equilibrium state is stable, if all the roots of the equation (7) are negative in case they are real or have negative real parts, in case they are complex.

4.1 Fully washed out state

In this case, we have

$$A\left(E_1\right) = \begin{bmatrix} -d_1 & 0 \\ 0 & g_2 \end{bmatrix} \quad (8)$$

The characteristic equation is $\left(\lambda + d_1\right)\left(\lambda - g_2\right) = 0$ (9)

The characteristic roots of (9) are $-d_1$ and g_2. Since one of these two roots is positive. Hence the fully washed out state is **unstable** and the solutions of the equations (5) are

$$u_1 = u_{10}e^{-d_1 t}; u_2 = u_{20}e^{g_2 t} \quad (10)$$

where u_{10}, u_{20} are the initial values of u_1, u_2 respectively.

Trajectories of perturbations:

The trajectories in the $u_1 - u_2$ plane are $\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{g_2}}$ (11)

4.2 Equilibrium state $E_2 : \bar{N}_1 = -e_1, \bar{N}_2 = 0$

In this state, the characteristic matrix is given by

$$A(E_2) = \begin{bmatrix} d_1 & 0 \\ 0 & g_2 \end{bmatrix} \quad (12)$$

The characteristic roots are d_1 and g_2. Since all the two roots are positive, hence the state is **unstable** and the equations (5) yield the solutions.

$$u_1 = u_{10} e^{d_1 t}; u_2 = u_{20} e^{g_2 t} \quad (13)$$

Trajectories of perturbations:

The trajectories in the $u_1 - u_2$ plane are $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{g_2}}$ (14)

4.3 Equilibrium state $E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2$

Here, we have $A(E_3) = \begin{bmatrix} -d_1 & 0 \\ 0 & -g_2 \end{bmatrix}$ (15)

The characteristic roots of (15) are $-d_1$ and $-g_2$. Since all the two roots are negative. Hence the state is **stable** and the solutions are

$$u_1 = u_{10} e^{-d_1 t}; u_2 = u_{20} e^{-g_2 t} \quad (16)$$

The trajectories of perturbations for this state are same as (14)

4.4 Equilibrium state $E_4 : \bar{N}_1 = -e_1, \bar{N}_2 = k_2$

The characteristic matrix is $A(E_4) = \begin{bmatrix} d_1 & 0 \\ 0 & -g_2 \end{bmatrix}$ (17)

The characteristic roots are d_1 and $-g_2$. Since one of these two roots is positive, hence the state is **unstable** and the equations (5) yield the solutions.

$$u_1 = u_{10} e^{d_1 t};\ u_2 = u_{20} e^{-g_2 t} \tag{18}$$

The trajectories of perturbations for normal steady state are same as the trajectories of fully washed out state.

5. NUMERICAL APPROACH:

The numerical solutions of the growth rate equations (1) and (2) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 4.

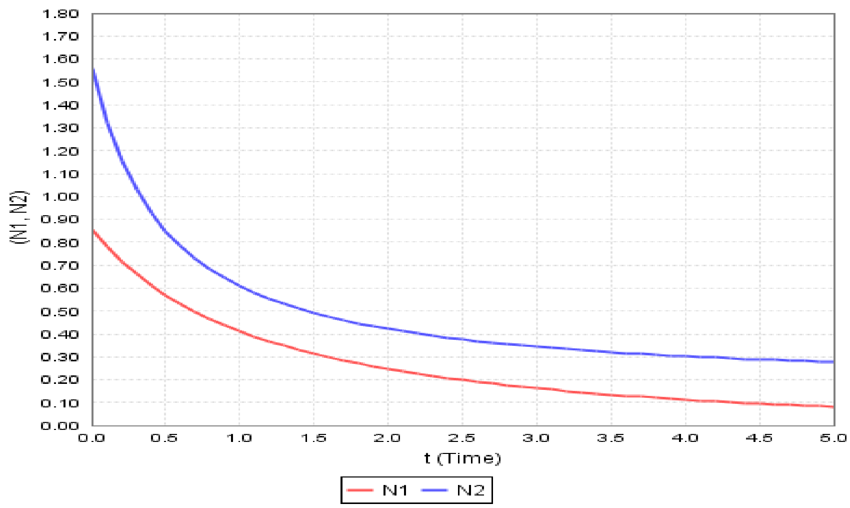

Figure 1. Variation of N_1 and N_2 against time (t) for d_1=0.783, g_2=1.152, a_{11}=0.333, a_{22}=0.45, N_{10}=0.621, N_{20}=0.963.

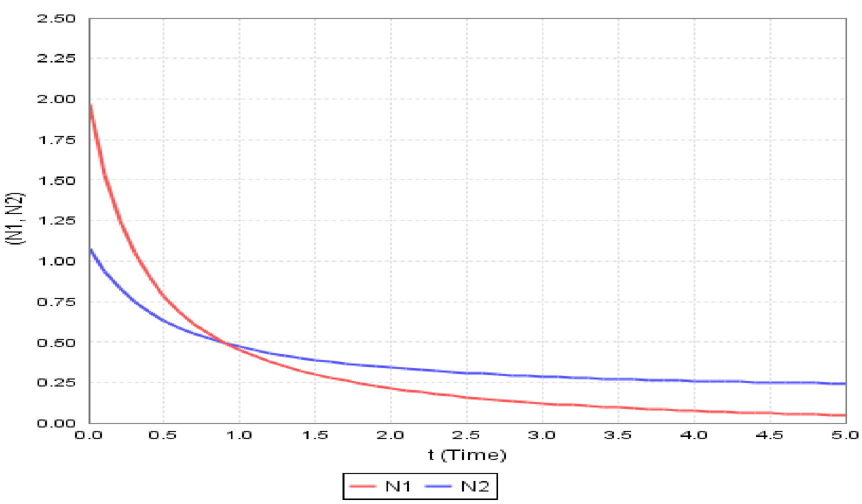

Figure 2. Variation of N_1 and N_2 against time (t) for d_1=0.9, g_2=0.756, a_{11}=0.585, a_{22}=0.9, N_{10}=1.971, N_{20}=0.792.

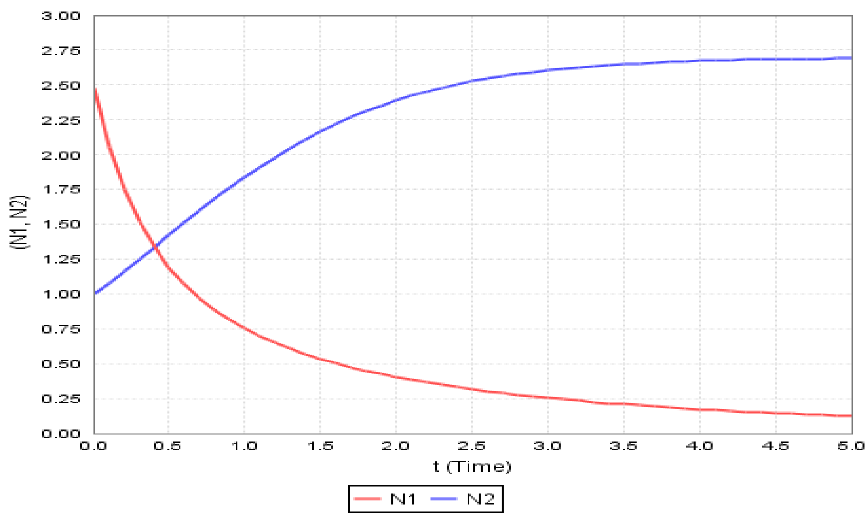

Figure 3. Variation of N_1 and N_2 against time (t) for d_1=0.225, g_2=1.287, a_{11}=0.729, a_{22}=0.477, N_{10}=2.475, N_{20}=0.999.

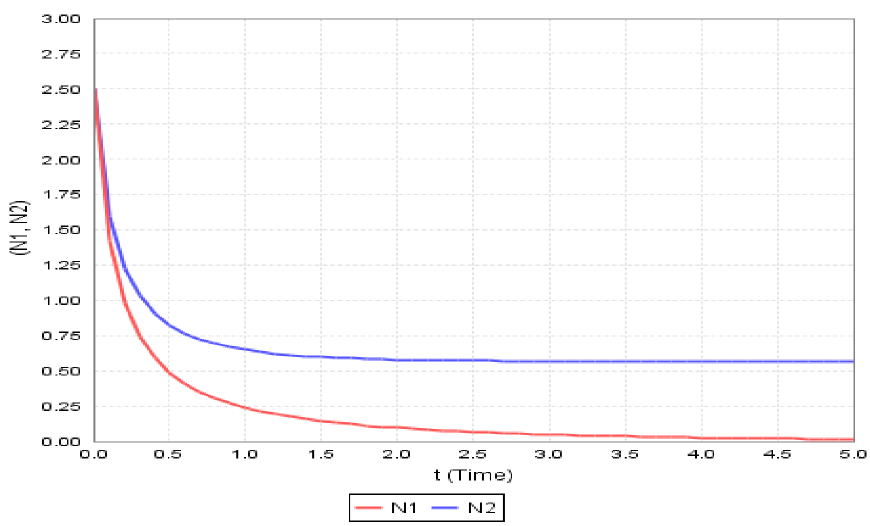

Figure 4. Variation of N_1 and N_2 against time (t) for d_1=0.477, g_2=1.818, a_{11}=2.745, a_{22}=3.195, N_{10}=2.5, N_{20}=2.5.

6. DISCUSSION & CONCLUSIONS:

In the present paper, we discussed on the two species model of neutralism with mortality rate for the first species. Here we established all possible four equilibrium states. It is observed that, in all equilibrium states, only one state E_3 is stable and rest of them unstable. Further, the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

Observations of the numerical solutions:

Case 1: In this case the two species decrease initially. The first species dominates over the second throughout. Further it is evident that both the two species asymptotically converge to the equilibrium point. This is illustrated in Figure 1.

Case 2: This is a situation at the self inhibition coefficient of S_2 and death rate of the first species are identical. In this case the initial conditions of S_1, S_2 are in decreasing order. The S_1 dominates over the S_2 initially up to the time $t^* = 0.92$ after which the dominance is reversed. (Figure 2).

Case 3: Initially the first species dominates over the second till the time instant $t^* = 0.41$ and thereafter the dominance is reversed. Further we notice that the second species has the least initial value. (Figure 3).

Case 4: In this case the initial conditions of the two species are identical. The death rate of the first species is less than of the growth rate of the second species. Further, in course of time we notice a steady variation with no appreciable growth rate in the second species. This is shown in Figure 4.

References

[1] A. Anna Sher and Jr. Molles, Manuel C , *Ecology: Concepts & Applications*, McGraw-Hill, New York, 2018.

[2] N. Arumugam, *Concepts of Ecology*, Saras Publication, India, 2006.

[3] M. Gillman, *An Introduction to Mathematical Models in Ecology and Evolution*, Wiley-Blackwell, U.K., 2009.

[4] M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, U.K, 2001.

[5] N.P. Kumar, *Some Mathematical Models of Ecological Commensalism*, Acharya Nagarjuna University, Ph.D. Thesis. 2010.

[6] Z. Ma, *The Study of Biology Model*, Anhui Education Press, China, 1996.

[7] M. Moghadas Seyed and Majid Jaberi-Douraki, *Mathematical Modelling,* John Wiley & Sons, New Jersey, United States, 2018.

[8] J.D. Murray*, Mathematical Biology*, Springer, New York, 1993.

[9] B.H. Prasad, Stability Analysis of a Three Species Non-Linear Eco-System with Restricted Resources, *The Journal of Indian Mathematical Society*, 2019, **86**: 126-136.

[10] B.H. Prasad, A Study on Discrete Model of a Prey-Predator Eco-System with Limited and Unlimited Resources, *The Journal of Indian Mathematical Society*, 2015, **82**: 169-179.

[11] B.H. Prasad, A Study on the Discrete Model of Three Species Syn-Eco-System with Unlimited Resources, *Journal of Applied Mathematics and Computational Mechanics*, 2015, **14** : 85-93.

[12] B.H. Prasad, A Study on Discrete Model of Three Species Syn-Eco-System with Limited Resources, *International Journal Modern Education and Computer Science*, 2014, **11**: 38-44.

[13] B.H. Prasad, A Discrete Model of a Typical Three Species Syn- Eco – System with Unlimited Resources for the First and Third Species, *Asian Academic Research Journal of Multidisciplinary*, 2014, **1(17)** : 36-46.

[14] B.H. Prasad and N. Ch. P. Ramacharyulu, Global Stability of Four Species Syn Eco-System: A Prey-Predator-Host-Commensal- A Numerical Approach, *Int. J. of Mathematics and Computer Applications Research*, 2013, **3**: 275-290.

[15] B.H. Prasad, On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts-III, *Journal of Experimental Sciences*, 2012, **3(2)** : 07-13.

[16] B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a typical three species syn eco-system, *Int. Journal of Mathematical Archive*, 2012, **3(10)** : 3583-3601.

[17] B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts – VII (Host of S2 washed out states), *Journal of Communication and Computer*, 2011, **8** : 415-421.

[18] B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts-V(Predator washed out states), *Int. Journal Open Problems Compt. Math.*, 2011, **4(3)** : 129-145.

[19] B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-II (Prey and predator washed out states), *Int. eJournal of Mathematics and Engineering*, 2010, **1(1)** : 60-74.

[20] P.D. Sharma, *Ecology and Environment*, Rastogi Publications, India, 2009.

[21] N.C. Srinivas, *Some Mathematical Aspects of Modeling in Bio-medical Sciences*, Kakatiya University, Ph.D Thesis, 1991.

[22] Sze-Bi Hsu, *Mathematical Modeling in Biological Science*, Tsing-Hua University, Taiwan, 2004.