THE CASE STUDY IN SURFACE AREA AND THE ARC LENGTH OF THE CURVE

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ABSTRACT:

This paper explains the concept of arc length of a curve and surface area using integration. The principle of integration is formulated by Isaac newton and Gottfried Wilhelm Leibniz. Definite integral is used to calculate the Arc length of a curve. The Surface area of revolution can be found using the arc length of a curve. Line segment is used for arc length which generates a Reimann sum.

KEYWORDS:

Definite integral, Surface revolution, change in horizontal segment, revolving line segment.

I. INTRODUCTION

A Rigorous Mathematical definition of integral was given by Bernhard Reimann. The Area under the curve and area between two curves is found using Definite integral. Definite integral is an integral expressed as the difference between the values of integrals at specified upper and lower limit of the independent variable. If x is definite integral then it is known as Reimann integral. Many real world application involve arc length.

Example: Rocket.

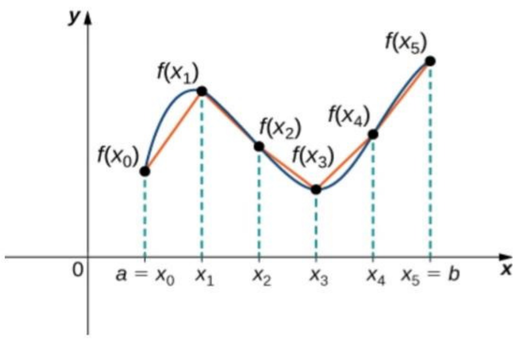
II. PROPERTIES OF DEFINITE INTEGRAL

- $\int_a^a f(x)dx = 0$. That is, if all of the Δx_i's are equal to 0, then the definite integral is 0.
- If f is integrable and f(x)≥0 on [a,b], then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval [a,b]. If f(x)≤0 on [a,b], then $\int_a^b f(x)dx$ equals the negative of the area of the region between the interval [a,b] and the graph of f.
- $\int_a^c f(x)dx+\int_c^b f(x)dx=\int_a^b f(x)dx$.
- If f and g are integrable on [a,b], then
  $\int_a^b (Af(x) + Bg(x))dx=$ A$\int_a^b f(x)dx + B\int_a^b g(x)dx$.
  For any constants A and B.
- If f is an odd function, then $\int_{-a}^a f(x)dx = 0$. That is, the definite integral of an odd function over a symmetric interval is zero.

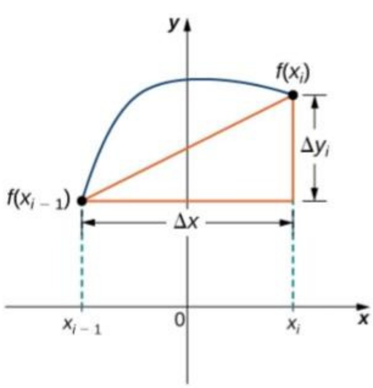
- If f is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

III. ARC LENGTH OF A CURVE Y=f(x)

The function f(x) to be integrable or at most continuous. We have a more stringent requirement for f(x) to calculate Arc length. Here f(x) to be differentiated and its derivative f'(x) is continuous. The functions which have continuous derivative is called smooth. Let f(x) be a smooth function defined over [a,b]. From the point (a,f(a)) to (b,f(b)) we have to calculate the length of the curve using line segment to approximate the length of curve we can start.



Let $p=x_i$ be a regular partition we construct a line segment from (x_i-1,f(x_i)-1) to point (x_i,f(x_i)) with n=5. To find the length of each line segment, we look at the change in vertical distance and also the change in horizontal distance at each interval. We have used regular partition. Δx give the change in horizontal distance at each interval. We useΔy_i=f(x_i)-f(x_i-1) to represent the change in vertical distance over the interval [x_i-1,x_i] as the change in vertical distance varies from interval to interval. At the interval [x_i-1,x_i] gives a representative line segment approximates the curve.



By Pythagoras theorem, the length of the line segment is

$\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

We can also write this as,

$$\Delta x\sqrt{1+((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean value theorem, there is a point $x_i^* \in [x_i\text{-}1, x_i]$ such that $f'(x_i^*)=(\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x\sqrt{1+[f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\,\Delta x.$$

This is a Reimann sum. Taking the limits as $n\to\infty$, we have

$$\text{Arc Length} = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$
$$= \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

IV. SURFACE AREA OF HEMISPHERE

The surface area of a Hemisphere can be easily found by the base of the sphere is circular in shape.Generally,there are two different types of surface area,they are total surface area and the curved surface area.

CURVED SURFACE AREA:

Curved surface area is defined by the area of the outer surface of the hemisphere.

TOTAL SURFACE AREA:

Total surface area is defined by the area of the curved surface and the area of the circle(base).

Since half of the sphere is hemisphere.

CSA of hemisphere =(1/2) surface area of the sphere

CSA=(1/2)4πr^2

CSA=$2\pi r^2$

The curved surface area of a hemisphere== $2\pi r^2$square units.

The total surface area of the sphere=curved surface area of sphere+base area

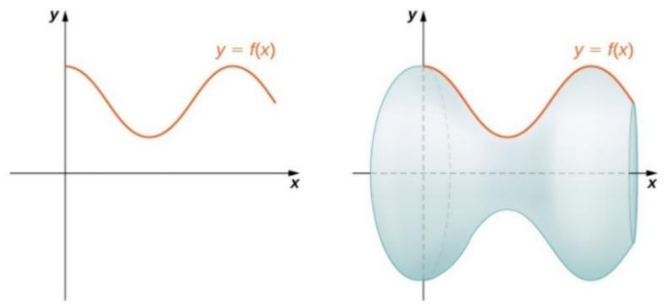
Hence,we know that the shape of the hemisphere is circular in shape,hence we use the area of the circle.

TSA=$2\pi^2+\pi r^2=3\pi r^2$

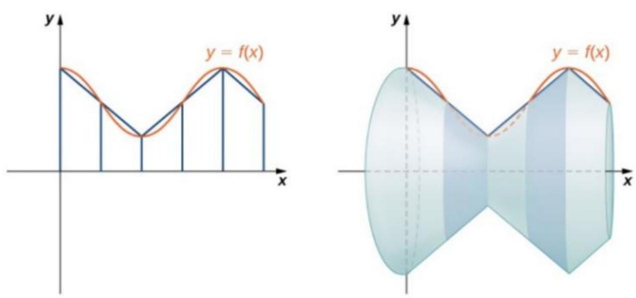
Hence,the total surface area of a hemisphere=$3\pi r^2$square units.

V. AREA OF SURFACE A SOLID

The Surface area of a solid is the area occupied by the solid. The Definition for surface area is sought by Henri Lebesgue and Hermann Minkowski. It was given in 12th century. The Surface should provide a function S→ $A(S)$ which assigns a positive real number to the surface. Additivity is the most fundamental property of surface area. The surface area of object is the sum of their faces. The situation is complex for curved surface.



The graph y=f(x) around x=axis is used to find the surface area of surface of revolution created by revolving. The function f(x) represents a curve and surface of revolution formed by revolving the graph of f(x) around x=axis. A band is produced where each line segment is revolved around the axis. They are the actually piece of cone.



$$\text{Surface Area} = \lim_{n\to\infty}\sum_{i=1} n^2\pi f(x_i^{**})$$

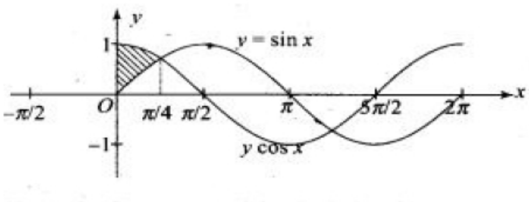
$$\Delta x\sqrt{1+(f'(x_i^*))^2}$$

$$= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})$$

VI. PROBLEMS

1. **Compute the area between the curve y= sin x and y= cos x and the lines x=0 and x=$\frac{\pi}{2}$**

Solution: To find the points of intersection solve the two equations.



Sin x= cos x= $\frac{1}{\sqrt{2}}$, Sin x= cos x=$\frac{1}{\sqrt{2}}$

X= $\frac{\pi}{4}$ X=$\frac{5\pi}{4}$

From the figure we see that cos x > sin x for 0 ≤ x <$\frac{\pi}{4}$ and sin x > cos x for $\frac{\pi}{4}$< x < π.

$$\text{Area} = \int_0^{\frac{\pi}{4}}(\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{4}}(\sin x - \cos x)dx$$

$$= (\sin x-(-\cos x))_0^{\frac{\pi}{4}} + (-\cos x - \sin x)_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= (\sin x + \cos x)_0^{\frac{\pi}{4}} + (-\cos x - \sin x)_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= [\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4} - \sin(0) - \cos(0)] + [-\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}]$$

$$= (\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 0 - 1) + (-0 - 1 + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}})$$

$$= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 1 - 1 + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}$$

$$= \tfrac{4}{\sqrt{2}} - 2$$

$$= 2\sqrt{2} - 2$$

$$= 2(1.414)-2$$

$$= 2.828-2$$

Area=0.828 sq.units.

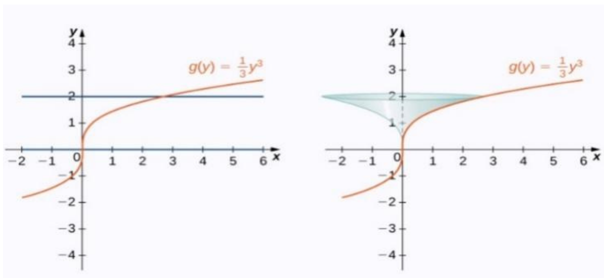
2. **Let f(x)=y=$\frac{1}{3}$(3x). Consider the portion of the curve where 0 ≤ y ≤ 2. Find the Surface Area of the Surface generated by revolving the graph of f(x) around the y-axis.**

Solution: Notice that we are revolving the curve around the y-axis, and the interval is in terms of y.

So, we want to write the function as a function of y.

We get x=g(y)=$\frac{1}{3}y^3$.

The graph of g(y) and the surface of rotation are shown in the following figure



The graph of g(y) and the Surface of revolution.

We have g(y)=$\frac{1}{3}y^3$.

so, g'(y)=y^2 and $(g'(y))^2$=y^4. Then,

Surface Area= $\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}$dy

$=\int_0^2 2\pi\,(\frac{1}{3}y^3)\sqrt{1+y^4}$dy

$=\frac{2\pi}{3}\int_0^2 (\,y^3\sqrt{1+y^4})$dy.

Let u= y^4+1. Then du= $4y^3$dy.

When y= 0, u= 1, and when y= 2, u= 17. Then,

$= \frac{2\pi}{3}\int_0^2 (\,y^3\sqrt{1+y^4})$dy

$= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}$ du

$=\frac{\pi}{6}[\frac{2}{3}u^{3/2}]|_1^{17}$

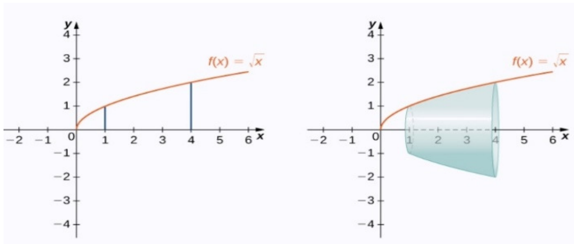
$=\frac{\pi}{9}[(17)^{3/2}-1]$

$\approx 24.118\ sq.units.$

3. **Let f(x)=$\sqrt{x}$ over the interval [1,4]. Find the Surface area of the Surface generated by revolving the graph of f(x) around the x-axis. Round the answer to three decimal places.**

Solution: The graph of f(x) and the Surface of rotation are shown in the following figure.

We have f(x)=$\sqrt{x}$.



The graph of f(x) and the surface of revolution.

Then f'(x)=1/(2$\sqrt{x}$) and (f'(x))^2=1/(4x). then,

Surface Area = $\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}$ dx

=$\int_1^4 \sqrt{2\pi\sqrt{x}1+\frac{1}{4x}}$ dx

=$\int_1^4 2\pi\sqrt{x+\frac{1}{4}}$ dx.

Let u=x+1/4. Then, du=dx.

When x=1, u=5/4, and when x=4, u=17/4. This gives us,

= $\int_0^1 2\pi\sqrt{x+\frac{1}{4}}$ dx

= $\int_{5/4}^{17/4} 2\pi\sqrt{u}$ du

= $2\pi[\frac{2}{3}u^{3/2}]\,|_{5/4}^{17/4}$

=$\frac{\pi}{6}[17\sqrt{17}-5\sqrt{5}]$

≈30.846 sq.units.

4.Find the length of the curve 24xy= y^4 + 48 from the point ($\frac{4}{3}$,2) to ($\frac{11}{4}$,4).

Solution: Solving for x in terms of y,

We get, x =$\frac{y^4+48}{24y}$ =$\frac{y^3}{24}$+$\frac{2}{y}$= g(y).

g'(y) = $\frac{3y^2}{24}$ -$\frac{2}{y^2}$

= $\frac{y^2}{8}$ -$\frac{2}{y^2}$.

$[g'(y)]^2 = \frac{y^4}{64} - \frac{4}{y^2} \cdot \frac{y^2}{8} - \frac{4}{y^4}$

$= \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}.$

$L = \int_c^d \sqrt{1 + [f'(y)]^2} dy$

$= \int_2^4 \sqrt{1 + \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}}\, dy.$

$= \int_2^4 \sqrt{\frac{y^4}{64} + \frac{1}{2} + \frac{4}{y^4}} dy$

$= \int_2^4 \sqrt{(\frac{y^4}{8} + \frac{2}{y^2})^2} dy$

$= \int_2^4 \left(\frac{y^4}{8} + \frac{2}{y^2}\right) dy$

$= \frac{y^3}{24} - \frac{2}{y} |_2^4$

$L = \frac{17}{6}.$

5.Find the surface area of a hemisphere whose radius is 4 cm?

Solution: Given:

Radius,r=4 cm

The curved surface area=$2\pi r^2$square units.

The total surface area=$3\pi r^2$square units

Substitute the value of r in the formula.

(1) CSA of the hemisphere=2×3.14×4×4

=3.14× 32

=100.48cm^2

(2)TSA of the hemisphere=3× $3.14 \times 4 \times 4$

TSA=3.14× 48

TSA=150.72 cm^2

Therefore, the total and curved surface area and total surface area of the hemisphere are 100.48 and 150.72 cm^2 respectively.

VII. CONCLUSION

The study on the concept of Arc length of a curve and Surface area makes us to understand the concept of definite integral in integration. This study tells about the Surface Revolution, total Surface, area, change in horizontal segment and change in vertical segment. Many real world application involve arc length.

VIII. REFERENCE

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